Visualization of the Road to Chaos for Finance and Economics Majors

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Efforts to simulate tur ulence in the financial mar ets include e eriments ith the lo istic e uation t t 1 1 t 1 , ith 0 t 1 and 0 d 4. Visual in esti ation of the lo istic e uation sho s the arious sta ilit and insta ilit re imes for the arious alues of the Fei en aum num er . Visualizations for t 20 o ser ations ro ide clear demonstrations of the sta ilit re imes. The author al e raicall anal zes all these re imes in more detail. For 0 3, the rocess settles to a uni ue sta le e uili rium. For 3 d 3.6, the rocess ifurcates, or, as colored isualization sho s ut not lac and hite, its itchfor ifurcation ranches an an s itch et een t o re imes. For 3.6 d 4, the rocess ecomes chaotic, i.e., deterministicall random. In this re ime are indo s of sta ilit , e. ., at

3 √2 3.8284. t 4, ure chaos, the rocess is e tremel sensiti e to initial alues, hich is clearl demonstrated isuall . The author increases the num er of o ser ations to t 1000, and com utes the homo eneous urst e onent of the rocess at 4 0.004, indicatin that t is lue noise, i.e., e tremel anti ersistent. histo ram sho s a hi hl lat urtic distri ution of t , ith an im loded mode , ith e tremel fat tails hi her than the mode , a ainst the reflectin alues at 0 and 1. e eral lots of the state director of the s stem in the t , t 1 s ace trace out the ara olic stran e attractor. lthou h the stran e attractor is a ell defined ara ole, the oints on the attractor set are deterministicall random and un redicta le.

Introduction

"Chaos theory", a modern development in mathematics and science, provides a framework for understanding irregular or turbulent fluctuations. Chaotic systems are found in many fields of science and engineering (Hall, 1991). The study of their dynamics is an essential part of the burgeoning science of complexity. Science of complexity researches the behavior of nonlinear dynamic processes, and has now reached to finance and economics (Savit, 1988), particularly in the form of a search for chaos in the financial markets after the stock market crash of October 19, 1987 (Hsieh, 1991, 1993; Patterson, 1993). The assessment by these analysts was clearly misdirected, since the stock market crash of October 19, 1987 was a discontinuity occurring in a persistent financial market,

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i.e., an example of market failure or inefficiency. Turbulence is a phenomenon that can only occur in antipersistent financial markets, i.e., in hyper-efficient markets like Foreign Exchange (FX) markets.

Historically, finance and economics have been cast in terms of linearized Newtonian physics. However, many phenomena in finance and economics are complex, nonlinear, self-organizing, adaptive, and feedback processes. Financial turbulence is conjectured to be a process to minimize friction between cash flows with different degrees of liquidity, with different investment horizons or with different trading speeds (Peters, 1994). Understanding these nonlinear processes is of importance to portfolio management, dynamic asset valuation, derivative pricing, hedging, trading strategies, asset allocation, risk management and the development of market neutral strategies.

Nonlinear dynamic processes are not new to financial economists. Mandelbrot (1963) found that speculative market prices followed a fractal differentiation process. More than 35 years later and using a different technology, Lo and MacKinlay (1999) come to the same conclusion. Moreover, nonlinear market dynamics had already been detected in high-frequency, intraday trading data by Müller, Dacorogna *et al.* (1990), and was recently confirmed with different data sets and different analytic techniques by Karuppiah and Los (2000).

Why should the study of nonlinear dynamic systems be of interest to financial economics? Because it offers a differentiated perspective on predictability in the financial markets. Financial processes can be differentiated according to their predictability. For example, Peters (1999, p. 164) discerns four cases of predictability (Table 1).

Table 1: Levels Of Predictability

Short-term		
High	Nonlinear Dynamic	Linear Dynamic
Low	Nonlinear Stochastic Dynamic	Complex Dynamic
Long-term	Low	High

While deterministic linear dynamic systems show high predictability, both in the short- and long-term, deterministic nonlinear dynamic systems show high predictability in the short-term, but low predictability in the long-term. Stochastic nonlinear systems, in general, show low predictability, both in the short- and the long-term. In contrast, complex systems show low short-term, but high long-term predictability.

The current financial-economic models of market pricing processes are often linear or linearized, but such models cannot differentiate between the various degrees of short- and long-term predictability. Linear models have high predictability both in the short- and long-term. In order to identify financial-economic models that differentiate between the short- and long-term predictability of pricing processes, one needs to introduce nonlinearity or complexity.

The particular research question of concern motivating this author is: since we find that financial market pricing processes are nonlinear, do they have high short-term and low long-term predictability, or are they complex, with low short-term, but high long-term predictability? For example, stock market pricing processes appear to have some short-term predictability, or persistence,[1] which is exploited by technical traders, but they are often

[1] Measured by Hurst exponents in the order of 0.6 - 0.7.

unpredictable in the longer-term, to the dismay of fundamental traders and investors (Savit, 1988). In addition, stock market return series show severe discontinuities, like the US stock market crises in 1929 and 1987, attesting to their persistence.

On the other hand, FX pricing processes are unpredictable, or anti-persistent,[2] in the short-term, but they tend to show some global predictability in the longer-term. For example, they appear to be rather resilient to exogenous shocks, like the Thai baht break 1997, or any other drastic revaluation. FX processes do not often show sharp discontinuities. In fact, the Thai baht break was exceptional and was probably caused by malfunctioning fundamental asset markets, e.g., of bank loans, in the Southeast Asian region. But they show intermittency: periods of stability and persistence are interrupted by periods of chaos.

In this paper, we simulate and analyze the properties of a particular complex nonlinear process in an effort to understand these various predictability regimes. We run simulation experiments with the logistic parabola. In particular, we observe four types of behavioral regimes generated by this model, depending on the value of its scaling parameter: (1) regimes of unique dynamic equilibria; (2) regimes of complex multiple dynamic equilibria; (3) regimes of intermittency, i.e., a mixture of multiple dynamic equilibria and chaos; and (4) complete chaos.

Visualization of the distributions of intermittency and of complete chaos, i.e., of nonstochastic, deterministically random behavior, and particularly the visualization of the chaotic attractor, produces several jarring surprises for conventional (probability and linearity based) statistics. Such experiments with the simple logistic parabola also convincingly demonstrate that complex behavior does not necessitate complex laws. Very simple nonlinear laws can produce very complex and unpredictable behavior.

Logistic Parabola

Let's start with the simple definition of a logistic dynamic process, where $x(t)$ may be the increments of a market price of a security (bond) or of an FX rate. The logistic parabola has been used to model restrained growth processes and has been applied in many fields, in particular in ecology and socioeconomics. We simulate, visualize and analyze its most salient features, in particular, its self-similarities generated by nonlinear iteration. We also compute Hurst exponents of its various stability regimes using wavelet multiresolution analysis.[3] We look at its stable and unstable regimes, the deterministic chaos it can produce, its bifurcation and phase shifting phenomena, its intermittency and the frequency distribution of chaos.

Definition 1: The logistic parabola is the following nonlinear differential equation:

$$x(t) = f(x)$$
$$= kx(t-1)\,[1 - x(t-1)]$$
$$= kx(t-1) - k\,[x(t-1)]^2, \text{ with } 0 \le x(t) < 1 \text{ and } 0 \le k \le 4 \qquad ...(1)$$

where k is a real number, for physical reasons.

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2 Measured by Hurst exponents in the order of 0.25 - 0.5.

3 The simulations with the logistic parabola were executed in a Microsoft £Excel 97 spread-sheet, on a Compaq ARMADA1700 notebook with Intel Pentium II processor. The growth parameter k was varied in steps of 0.1 for $t = 1,2,..., 1,100$. The Hurst exponents based on wavelet multiresolutuion analysis of level 3 of the simulated data using the software Benoit, Version 1.2, of TruSoft Int'l Inc. 1997, 1999.

Remark 1: This logistic parabola, or **quadratic map**, was introduced in 1845 by the Belgian sociologist and mathematician Pierre-Francois Verhulst (1804-1849) to model the growth of populations limited by finite resources (Verhulst, 1845). The designation "logistic", however, did not come into general use until 1875. It is derived from the French *logistique*, referring to the "lodgment" of troops. Interesting details of this logistic process, particularly about its strange attractor set, can be found in Schroeder (1991). Notice that there is no harm in assuming that the variable x is suitably scaled so as to lie between zero and one.

The logistic parabola is an extremely simple nonlinear differential equation, which consists of a linear part, $kx(t - 1)$, and a nonlinear part, $- k[x(t - 1)]^2$. It exhibits stable, bifurcating, intermittent and completely chaotic process regimes for certain values of the scaling parameter k, caused by its implied iterative, binomial "folding" process. The process can swing from stable behavior to intermittent behavior, and then back to chaotic behavior, by relatively small changes in the value of its single scaling or growth parameter k (Feigenbaum, 1979). This scaling parameter k governs the transitions between the various stability regimes of this nonlinear dynamic feedback process.

Stability and Persistence Regimes

These various process regimes are summarized by the Feigenbaum diagram, or "fig tree" plot in Figure 1, which shows the steady state equilibrium values of $x(t)$ in the observation range $t = 101, \ldots, 200$ for various values of the scaling parameter k, which is the sole control parameter of the logistic process.

Figure 1: Bifurcation Diagram of Logistic Parabola

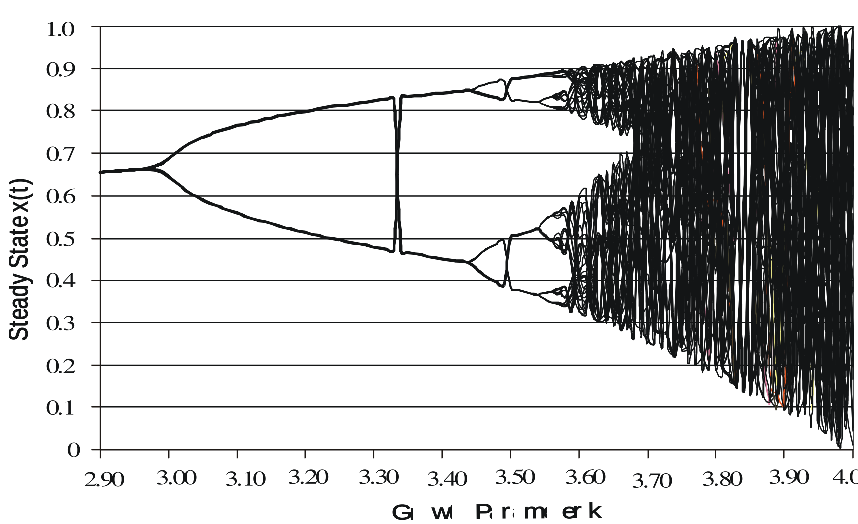
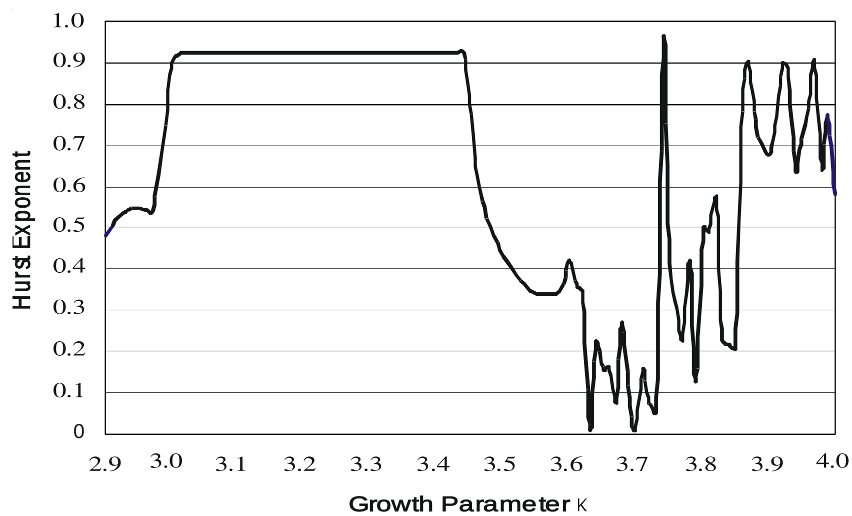


Figure 2 provides the corresponding Hurst exponents, computed from three resolution levels of wavelet coefficients, indicating the relative persistence of the logistic process for various values of k. For $H = 0.5$ the process is white noise or non-persistent; for $0 < H < 0.5$ the process is antipersistent; for $0.5 < H < 1$ the process is persistent.

Figure 2: Hurst Exponents of Logistic Process x_k (t)



Notice in Figure 1 the unique stationary, timeless, homogeneous states of equilibrium for $k < 3$, and the apparent multiplicity of equilibrium states after the critical value of $k = 3.0$. There is a cascade of supercritical, period-doubling pitchfork bifurcations and phase-shifting crossovers for $3 < k < 3.6$, followed by moderate chaos. The first bifurcation at $k = 3.0$. But the coloration in Figure 1 clearly shows that there are crossovers, or 180 degree phase shifts, in the process paths, signifying the occurrence of bistability along particular paths. The first crossover occurs at $k = 3.34$. This is followed by a set of bifurcations at $k = 3.45$, followed by a crossover switch at $k = 3.50$, etc.

In this regime the steady state of $x(t)$ is strictly unpredictable, because it depends on the initial state value and the precision of the computations (= computation noise), but it can be characterized by one or the other path. This dependence of the system on different steady state paths according to past history is called "hysteresis".[4]

Notice in Figure 2, that sharp changes in the Hurst exponent do detect the bifurcations. At $k = 3.0$ the simple symmetry of the steady state equilibrium is broken. Between $k = 3.0$ and $k = 3.45$ the Hurst exponent is homogeneous $H = 0.924$, indicating great persistence, because of the resulting bi-stability. At $k = 3.45$ many more bifurcations appear, and at $k = 3.6$ chaos appears. The Hurst exponent drops sharply in both cases. Obviously, the Hurst exponent, which measures the relative persistence of a process, does not detect the phase shifts in the process at $k = 3.34$ and $k = 3.50$.

Chaos, which is unpredictable deterministic evolutionary behavior, appears in the range $3.6 < k < 4$. Here the sharp classical distinction between chance and necessity, between stochastic and deterministic behavior is blurred and we witness the emergence of complexity. It appears that, in this particular range of k, there exist deterministic randomness. In this range

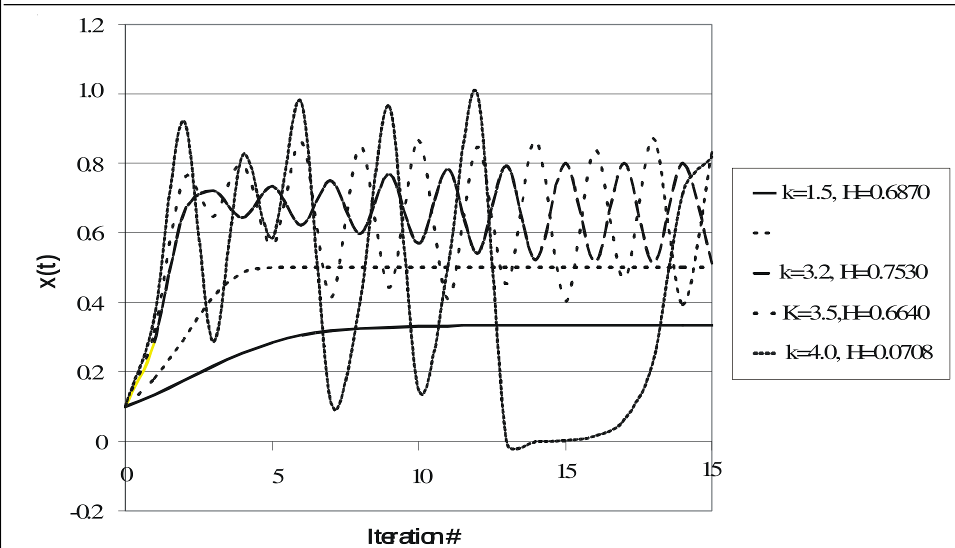
4 The stability of differential maps which map the unit interval into itself is discussed in greater detail in Singer (1978).

the Hurst exponent is heterogeneous, indicating the multi-fractality of the logistic process. Between $k = 3.6$ and $k = 3.74$, $0 < H < 0.3$ and the process is very antipersistent. Then, at $k = 3.74$, the process is suddenly very persistent, but immediately thereafter, between $k = 3.74$ and $k = 3.83$, $0.3 < H < 0.6$, the process is only moderately anti-persistent. This moderate persistence is similar to what is observed in the FX markets.

In particular, this moderate chaos regime is interleaved at certain values of k with periodic "windows" of relative calm. The most prominent being the 3-period window starting at $k = 1 \quad 2\sqrt{2} = 3.83$. Once the period length 3 has been observed, all possible periods and frequencies appear and complete deterministic chaos results (Li and Yorke, 1975). Interestingly, after $k = 3.83$, $0.6 < H < 0.9$, chaos has become, counter-intuitively, persistent, because of the emergence of "structure," and "self-organization." However, on very close detailed observation, within the 3-period window period-doubling reappear, leading to stable orbits of period length $3 \iota 2 = 6$, $3 \iota 2^2 = 12$, $3 \iota 2^3 = 24$, etc., and renewed chaos, in which another 3-period window is embedded, and so on, *ad infinitum* into other *self-similar cascades of orbits* of period length 3.2^n. This is the process regime of turbulence. The cascades of orbits form one-dimensional "eddies".[5]

We'll now discuss each of the four regimes of evolutionary behavior of the logistic parabolic process in a cursory fashion at low values of t to see how quickly the process stabilizes. Notice the changes in the behavior of $x(t)$, by looking at its first 20 iterations, $t = 1,..., 20$, for various values of the scaling parameter k, starting at $x(0) = 0.1$ (Figure 3).[6]

Figure 3: Convergence and Period-Doubling



5 For more images of critical points of nonlinear dynamical mappings, see Jensen and Myer (1985).

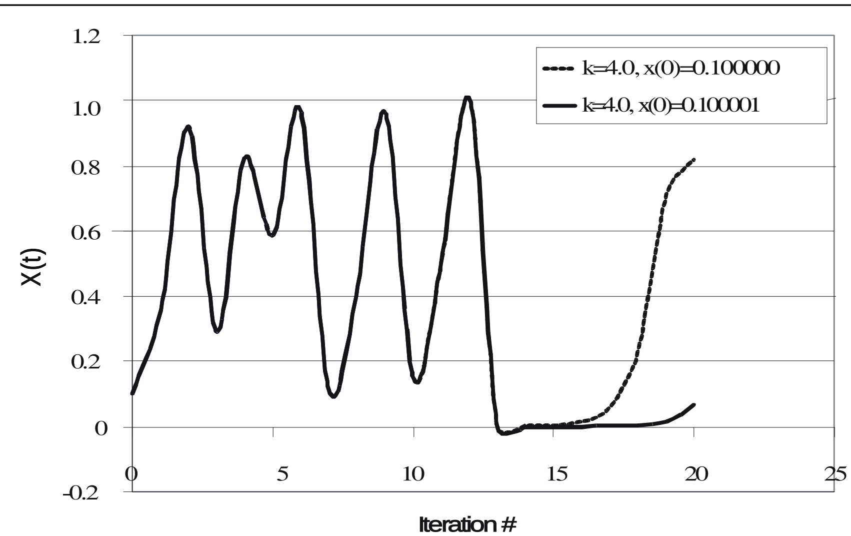
6 Notice that the Excel spreadsheet plotter has a problem with sharp discontinuities at the bottom of the chaotic, $k = 4.0$ process. Excel's spline smoother curves the line below $x(t) = 0$, although the process $x(t)$ t 0, always.

When k = 1.5, $x(t)$ reaches its steady state of $x^* = \frac{1}{3}$ at about t = 8. When k = 2.0, $x(t)$ reaches its steady state of $x^* = \frac{1}{2}$ at about t = 4. In these regimes of unique uniform steady states (which are also asymptotically stable), the system ignores time. Once it has reached a steady state, it does not matter where we are in time: the value of the system remains one and the same for each t. These regimes thus uniquely Newtonian and stationary. But for k = 3.2, the system produces oscillations between two steady states. First, it appears to settle in a periodic rhythm by about t = 16. Now the system is clearly time dependent: it differs in value depending on the phase of the periodicity. For about k = 3.5, there appear to be two different periodicities superimposed on each other and thus oscillations between four different steady states. For k = 4, any specific periodicity has completely vanished, although there are still nonperiodic cyclical oscillations.

The Hurst exponent, which ranges between 0 and 1, computed for these first 20 observations, is H t 0.664 > 0.5 for k = 1.5, 2.0, 3.2, and 3.5, indicates that these particular logistic processes are persistent or pink, i.e., between white and brown noise. But H = 0.07 < 0.5 for k = 4.0, indicating that this chaotic process is initially antipersistent.

The completely chaotic process at k = 4.0 is very unstable: the logistic process is extremely sensitive to the initial condition, i.e., to the starting point of the process $x(0)$. Small changes in the initial condition lead to large amplifications of the effects of these changes. In Figure 4, we show two paths for $x(t)$ for when $z(0)$ = 0.100000 and when $x(0)$ = 0.100001, with a small change in the sixth position after the decimal point.

Figure 4: Sensitivity to x(0)



Notice that until the two process traces split, they are exactly the same: their maxima and minima follow in exactly the same order at the same time. But at the end, the temporal symmetry of the steady state solutions is broken: the equilibrium has become time dependent.

Meteorological processes are often considered to show regime changes from stable to chaotic. An even simpler example of such a regime change is cigarette smoke. When it arises from a cigarette, the smoke is first a smooth stable laminar flow, until it rather suddenly becomes a chaotic "whirl."

We will now discuss the four regimes in more detail, visualize them and algebraically analyze when and why they occur.

Steady State Solutions

Since we observed that the logistic process stabilizes rather quickly to its steady states, in the following only its equilibria are analyzed, which are dependent on the scaling parameter k. For values of $k < 3.0$, the logistic process settles to a unique static equilibrium are:

Definition 2: The *static equilibrium* or *steady state solution* is reached when

$$x(t) = x(t-1) = x^*, \text{ a constant} \quad ...(2)$$

Example 1: For the 1-orbit, from solving the not iterated logistic equation,

$$kx^*(1-x^*) = kx^* - k(x^*)^2 = x^* \quad ...(3)$$

for a nontrivial solution

$$x^* = \frac{k-1}{k} \quad ...(4)$$

The slope of the logistic parabola is

$$\frac{\partial x(t)}{\partial x(t-1)} = k[1-2x(t-1)] \quad ...(5)$$

which equals k for $x(t-1) = 0$ and $2-k$ for the unique steady state solution $x^* = \frac{k-1}{k}$.[7] This dynamic equilibrium is stable as long as the slope of the logistic parabola

$$\left|\frac{\partial x(t)}{\partial x(t-1)}\right| = |k[1-2x(t-1)]| < 1 \quad ...(6)$$

Thus the single point $x(t-1) = x^* = 0$ is stable for $0 \leq k < 1$ and marginally stable for $k = 1$, but it is unstable for $1 < k$.

The second steady state solution $x^* = \frac{k-1}{k}$ exists and is stable for $1 < k < 3$, because then

$$\left|\frac{\partial x(t)}{\partial x(t-1)}\right| < 1 \quad ...(7)$$

[7] This is a familiar result for economists who have studied Solow's one-period delayed stable market pricing spiral, or dynamic cobweb model, towards a unique equilibrium.

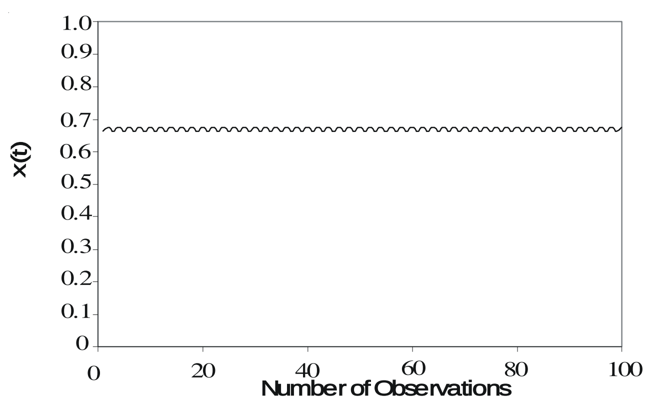
For example when $k = 1.5$, $x^* = \frac{1}{3}$, and $\left|\frac{\partial x(t)}{\partial x(t-1)}\right| = 0.5$. When $k = 2.0$, $x^* = \frac{1}{2}$, and $\left|\frac{\partial x(t)}{\partial x(t-1)}\right| = 0$. The steady state $x^* = \frac{1}{2}$ is always superstable (with period length 1), and convergence to this particular state is always very rapid.

However, something happens when $k = 3$. At $k = 3$, the steady state is $x^* = \frac{2}{3}$ but the slope of the logistic parabola is $\left|\frac{\partial x(t)}{\partial x(t-1)}\right| = 1$ and the process no longer converges (= stably attracted) to x^*! This steady state is marginally stable nearby values of $x(t)$ are not attracted to nor repelled from $x^* = \frac{2}{3}$.

Self-organization: Period Doubling

At $k = 3$, two possible steady state solutions x^* appear, very closely together, but clearly separated, between which the process $x(t)$ alternates (Figure 5).[8] The process remains very predictable, since the oscillation between the two stable states is regular. When $x(t)$ is shocked at this value of $k = 3$, it still quickly returns to this oscillation sequence. This event for $k = 3$ is called a (Myrberg) bifurcation or period doubling

Figure 5: Two Stable Equilibria (k-3)



Let's first analyze the regime with two steady state equilibria. An cyclical trajectory, or orbit, having a period length of $P = 2$, or 2-orbit, is the steady state solution x^*, which satisfies the 1 ι iterated logistic equation:

$$f(f(x)) = k\{kx^*(1-x^*)[1-kx^*(1-x^*)]\} = x^* \quad ...(8)$$

[8] Figure 5 and the following Figures portray $x(t)$ for $t = 901 - 1000$, after the process has completely stabilized.

The value of k for the superstable steady state solutions $x^* = 0.5$ is obtained from solving the once-iterated logistic equation

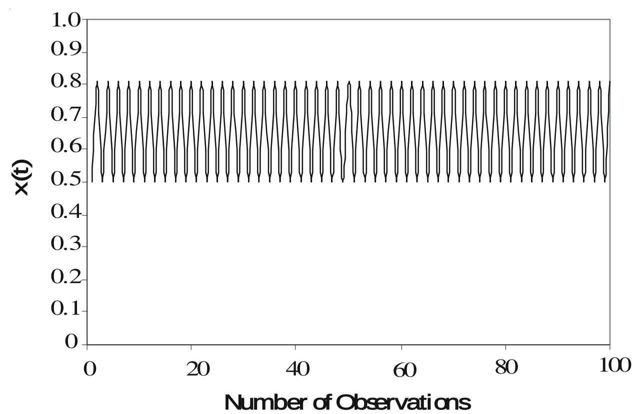
$$k\{0.5k(1-0.5)[1-0.5k(1-0.5)]\} = 0.5 \qquad ...(9)$$

or

$$k^3 - 4k^2 + 8 = 0 \qquad ...(10)$$

which has three solutions for the scaling parameter: $k = 2$, corresponding to $x^* = 0.5$, $k = 1 + \sqrt{5} = 3.2361$, corresponding to $x^* = 0.8090$[9], and the inadmissible solution $k = 1 - \sqrt{5} = -1.2361 < 0$. Thus, for $k = 2$ and $k = 1 + \sqrt{5}$, respectively, there are two stable steady states or frequencies, i.e., two alternating, stable orbits of period length $P = 2$. Accordingly, $x(t)$ takes on the values $x(0) = 0.5 \rightarrow x(1) = 0.8090 \rightarrow x(2) = x(0) = 0.5 \rightarrow x(3) = x(1) = 0.8090$, etc. as seen in Figure 6.

Figure 6: Two Stable Equilibria (k-3.24)



When k is increased further, then these two steady states of the once-iterated logistic parabola will in turn become unstable at precisely one and the same value of k. This is not a coincidence since, according to the chain rule of differentiation:

$$\frac{\partial}{\partial x}[f(f(x))]_{x=x(0)} = [f'(f(x))]_{x=x(0)} \cdot [f'(x)]_{x=x(0)} = f'(x(1)) \cdot f'(x(0)) \qquad ...(11)$$

When $x(0)$ becomes unstable, because

$$\left|\frac{\partial}{\partial x}[f(f(x))]_{x=x(0)}\right| > 1 \qquad ...(12)$$

[9] Solve the once iterated equation $(1+\sqrt{5})\{(1+\sqrt{5})x^*(1-x^*)[1-(1+\sqrt{5})x^*(1-x^*)]\} = x^*$ for x^*. The growth parameter $k = 1 + \sqrt{5} = 3.2361 = \frac{2}{\gamma}$, where $\gamma = 0.618....$, i.e., the golden mean.

so does x(l) at the same vale of k. Thus, both these steady state solutions of the once-iterated logistic equation f(f(x)) will bifurcate at the same k value, leading to an orbit of period length $P = 2^n = 2^2 = 4$. In other words, f(f(f(f(x)))) will have n = 4 steady state solutions or frequencies. The 4-orbit has four consecutive steady state solutions x*, which satisfies the 3 ι iterated logistic equation.

$$f(f(f(f(x^*))))=x^* \qquad ...(13)$$

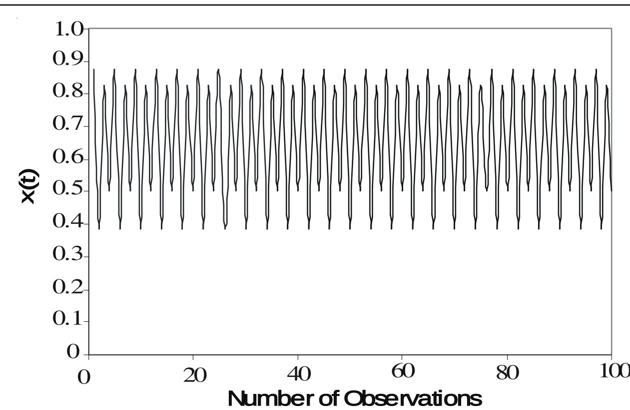
Again, the value of k = 3.4985 for superstable steady state solutions x* with a 4-orbit is obtained from solving the 3 ι iterated logistic equation

$$f(f(f(f(0.5))))=0.5 \qquad ...(14)$$

Accordingly x(t) produces the superstable orbit of period length 4 of f(x): x(0) = 0.5 O x(l) = 0.875 O x(2) = 0.383 O x(3) = 0.827 O x(4) = x(0) = 0.5, etc., as seen in Figure 7.

Again, because of the chain rule of differentiation, the four derivatives are the same at all four points of the orbit. Thus if, for a given value of k, the magnitude of one of the derivatives exceeds 1, then the magnitude of each of the four will. Hence, all four iterated x(t) will

Figure 7: Four Stable Equilibria (k=3.50)



bifurcate at the same value of k, leading to a cyclical trajectory, or orbit, of period length $P = 2^n = 2^3 = 8$, etc.

In summary, the general method for finding the value of the scaling parameter k for which a superstable orbit with period length P exists, is to solve the equation

$$f^{(P)}(0.5) = 0.5 \qquad ...(15)$$

exactly for k, where P is the period length of the orbit and $f^{(P)}$ is the $(P-1)^{th}$ iteration of the steady state logistic parabola

$$f(x^*) = kx^*(1-x^*) \qquad ...(16)$$

Self-similarity and Scaling

The period-doubling transformation of the logistic parabola is asymptotically self-similar. Feigenbaum (1979) proved that it obeys a scaling law with the following scaling factor:

$$\alpha^{(n)} \xrightarrow[n\to\infty]{} \frac{x^{(n)}_{P/2} - x(0)}{x^{(n+1)}_{P/2} - x(0)} \to -2.5029... = \alpha \qquad ...(17)$$

where $x^{(n)}_{P/2}$ is the value of the iterate x at the half period $\frac{P}{2}$ for a superstable orbit of period length $P = 2^n$, with $n = \log_2(P)$, starting with $x(0) = 0.5$. This scaling factor is related to Feigenbaum's universal constant δ which appears in the following geometric law of the scaling parameter k[10]:

$$\delta_n \xrightarrow[n\to\infty]{} \frac{k^{(n)} - k^{(n-1)}}{k^{(n+1)} - k^{(n)}} \to \delta = 4.6692016091029... \qquad ...(18)$$

Feigenbaum also discusses a simplified theory, which yields the following relationship between the scaling factor α of the scaling parameter scaling law and the universal constant δ

$$\delta \approx \alpha^2 + \alpha + 1 \approx 4.76 \qquad ...(19)$$

Spectral Analysis of Periodic Orbits

Let $c^{(n)}_k$ be the Fourier coefficient of the $x^{(n)}(t)$ for a period length $P = 2^n$. In going from an orbit of period length $P = 2^n$ by a period-doubling bifurcation to an orbit of period length $P = 2^{n+1}$, the new Fourier coefficients with an even index $c^{(n+1)}_{2k}$, which describe the harmonics or periodicities of the regular orbits, are approximately equal to the old Fourier coefficients:

$$c^{(n+1)}_{2k} \approx c^{(n)}_k \qquad ...(20)$$

because periodicity causes

$$x^{(n+P)}(t) \approx x^{(n)}(t) \qquad ...(21)$$

The odd-indexed Fourier coefficients $c^{(n+1)}_{2k+1}$, which describe the subharmonics appearing in the spectrum as a result of period doubling, are determined by the difference

$$x^{(n+P)}(t) - x^{(n)}(t) \qquad ...(22)$$

It can be shown (Feigenbaum, 1979) that the squared magnitudes, or power ratios, of these odd-indexed Fourier coefficients, $\left|c^{(n+1)}_{2k+1}\right|^2$, are roughly equal to an adjacent component from the previous orbit scaled down by a factor of

[10] This Feigenbaum constant δ was originally found by Grossman and Thomae (1977).

$$\frac{8D^4}{(1\ \ D^2)}\quad \frac{8(\ 2.5029)^4}{1\ \ (\ 2.5029)^2}\quad 43.217 \qquad \text{...(23)}$$

corresponding to

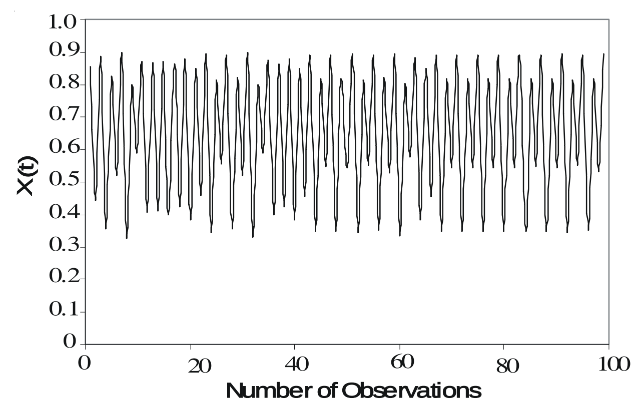
$$10\ \log_{10} 43.217 = 16.357\text{dB} | 16\text{dB} \qquad \text{...(24)}$$

where dB=decibels.[11] When the scaling parameter k is increased, more and more subharmonics appear until deterministic chaos or noise is reached, as we will see in the following sections.

Intermittency and Chaos

The bifurcation scenario repeats itself as k is increased, yielding orbits of period length 32, 64, etc., ad infinitum, until at about k = 3.6, this dynamic process appears to become unstable. The process ends up in an undefined orbit of infinite period length, of which Figure 8 gives only a sample "window" of 100 observations.

Figure 8: Unstable Equilibria (k=3.60)



The cyclical trajectory or orbit is now aperiodic, comprising a strange point set of infinitely many values of x(t) that never precisely repeat, although there is cyclicity. The approximate self-similarity of this point set shows Feigenbaum's self-similarity scaling factor of about L = -2.5029. The Hausdorff dimension D = 0.538...of this point set, which is a Cantor set, was derived analytically and numerically by Grassberger (1981). A good approximation is:

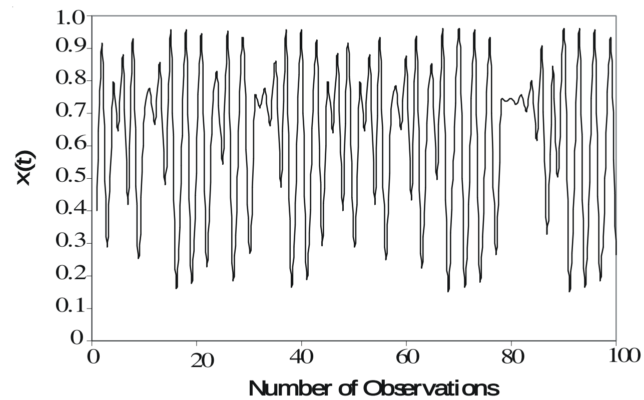
$$D\ \ \frac{\log J}{\log \frac{1}{2.5}}\ | 0.525 \qquad \text{...(25)}$$

where the golden mean $J\ \ \frac{\sqrt{5}\ \ 1}{2}$ = 0.61803....Thus this trace is almost half way in between a line (D = 1) and a set of points (D = 0), with a slight balance in favor of a line.

[11] The number of decibels is, by definition, $10\log_{10}$ of a squared magnitude ratio, such as the spectral power ratio.

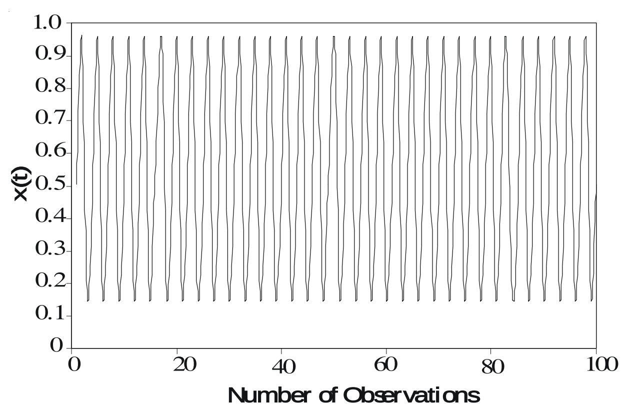
Intermittency: However, within the chaotic region, when $3.6 \leq k < 4.0$, some "windows of stability" do occur in between periods of chaos. This alternation of stability and chaos when k is increased is called *intermittency*. For example, for $k = 3.82$, we have moderate chaos (Figure 9).

Figure 9: Moderate Chaos (k=3.82)



But then, stability appears to reappear at $k = 1+\sqrt{8} = 1+\sqrt{2^3} = 1+2\sqrt{2} = 3.83$. There is the so-called "tangent bifurcation" at this value $k = 3.83$ (Figure 10). This is also indicated by the Hurst exponent in Figure 2. It looks as if for $k = 3.83$, there are only two stable equilibria $x^* = 0.154$ and $x^* = 0.958$, but the process $x(t)$ passes through the nonattracting, marginally stable equilibrium $x^* = 0.5$.

Figure 10: Three Stable Equilibria (k = 3.83)



Just above $k = 3.83$, the thrice-iterated logistic parabola acquires six additional steady state points x^*: three with an absolute slope $\left|\frac{\partial f^{(3)}(x(t))}{\partial x(t-1)}\right| > 1$ which belong to the unstable orbit of period length 3, and three with a slope $\left|\frac{\partial f^{(3)}(x(t))}{\partial x(t-1)}\right| > 1$, which are the three points belonging to the stable orbit with period length 3 and the apparent periodicity breaks down (Figures 11 and 12).

Figure 11: Three Stable + Three Unstable Equillbria (k=3.85)

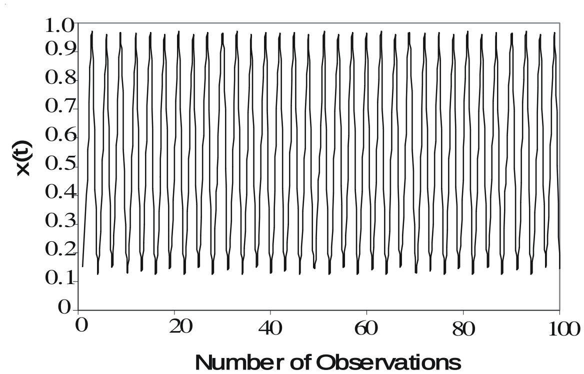
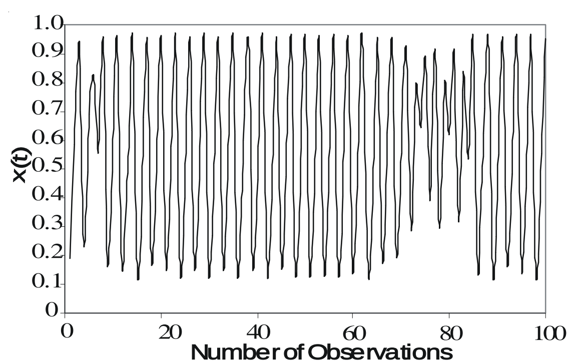


Figure 12: Intermittency (k=3.86)



Thus we encounter the famous period-3 orbit, an orbit with three distinct frequencies, which guarantees that all other period lengths or frequencies exist, albeit as unstable orbits, at the same parameter value.[12] In other words, the twice-iterated process.

$$f(f(f(x^*)=x^* \qquad ...(26)$$

has a 3-orbit with three consecutive steady state solutions x^*, which satisfies the 2x iterated logistic parabola:

[12] In 1971 a Belgian physicist, David Ruelle, and a Dutch mathematician, Floris Takens, together predicted that the transition to chaotic turbulence in a moving fluid would take place at a well-defined critical value of the fluid's velocity. They predicted that this transition to turbulence would occur after the system had developed oscillations with at least three distinct frequencies. Experiments with rotating fluid flows conducted by American physicists Jerry Gollub and Harry Swinney in the mid-1970s supported these predictions.

$$f(f(f(0.5))) = k\left\{k\{0.5k(1-0.5)[1-0.5k(1-0.5)]\}\left(1-k\{0.5k(1-0.5)[1-0.5k(1-0.5)]\}\right)\right\} \qquad ...(27)$$

$$= \frac{k^3}{4}\left(\frac{4-k}{4}\right)\left[1-\frac{k^2}{4}\left(\frac{4-k}{4}\right)\right] = 0.5$$

or

$$k^3(4-k)(16-4k^2+k^3) - 128 = 0 \qquad ...(28)$$

which has seven exact solutions, three of which are real and four of which are conjugate complex. Numerically, these solutions are:[13]

$k = 2$, which corresponds with the superstable equilibrium $x^* = 0.5$.

$k = 3.832$, which corresponds with the equilibria $x^* = 0$ (marginally stable), $x^* = 0.154$ (stable), $x^* = .0.165$ (unstable), $x^* = 0.499$ (stable), $x^* = 0.529$ (unstable), $x^* = 0.739$ (stable), $x^* = 0.955$ (unstable), $x^* = 0.958$.

$k = -1.832$, which is inadmissible, because $0 < k$.

$k = 2.553 +.959i$ and $2.553 -.959i$, which are inadmissible, because k is real.

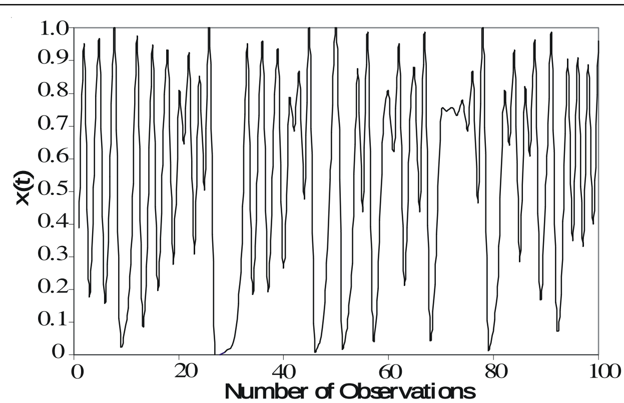
$k = -.553 +.959i$ and $-.553 -.959i$, which are inadmissible, because k is real.

The eight equilibria corresponding to $k = 3.832$ are found from the equation:

$$3.832\left[3.832(3.832x^*)(1-3.832x^*(1-x^*))\right]\left[1-3.832(3.832x^*)(1-x^*)(1-3.832x^*(1-x^*))\right] - x^* = 0 \qquad ..(29)$$

After period length 3 has appeared at $k = 3.83$, orbits of any period length are possible. As Li and Yorke (1975) state "period three implies chaos." Finally, at $k = 4.0$ we encounter complete chaos (Figure 13). Chaos is the coexistence of an infinite number of unstable deterministic orbits.

Figure 13: Complete Chaos (k=4.0)



[13] Obtained with Maple symbolic algebra software in Scientific Workplace, Version 3.0.

Universal Order of Period Lengths

The reason for the appearance of any period length, or frequency, after period length 3 is that the different period lengths P of stable periodic orbits of uni-modal maps, like the logistic parabola, do not appear randomly. In fact they appear in a universal order, as proved by Sharkovskii (1964):

If k_p is the value of the scaling parameter k at which a stable period of length P first appears, as k is increased, then $k_p > K_q$ tfp q(read: p precedes q) in the following Sharkovskii order:

$$
\begin{array}{l}
3 \quad 5 \quad 7 \quad 9 \quad \ldots \\
2.3 \quad 2.5 \quad 2.7 \quad \ldots \\
\ldots \\
2^n.3 \quad 2^n.5 \quad 2^n.7 \quad \ldots \\
\ldots \quad 2^m \quad \ldots \quad 2^2 \quad 2 \quad 1
\end{array}
\qquad \ldots(30)
$$

Example 2: The minimal k value for an orbit with p = 10 = 2-5 is larger than the minimal k value for $p = 12 = 2^2 \bullet 3$ because period length 10 12.

Remark 2: (1) Thus the existence of period length p = 3 guarantees the existence of any other period length q for some $k_q < k_p$; (2) if only a finite number of period lengths occur, their length must be powers of 2; and (3) if a period length p exists that is not a power of 2, then there are infinitely many different periods.

Interestingly, the intervals of k for the stable orbits are dense. That implies that the parameter values for which no stable periodic orbits exist form no intervals. Nevertheless, they have a positive Lebesgue measure. This means that a random choice of the scaling parameter k has a nonvanishing probability of leading to an aperiodic orbit. These aperiodic orbits are thus not “unlikely.” They have a particular probability of occurrence, although that probability may be very small.

Complete Chaos

With k = 4, the process has become completely chaotic. In Figure 14 we look at the ultimate chaotic pattern of the logistic x(t) for k = 4.0 and t = 101,..., 1100. The Hurst exponent H = 0.58, indicating that this logistic chaos exhibits some persistence and is not completely white. No random number generator is used! The deterministic logistic parabola generates these 1000 values of x(t), after the first 100 values were discarded, starting from a;(0) = 0. The process x(t) has a bounded range: 0 < x(t) < 1, but its mean is undefined, as is its variance, no matter how many observations generated. Ergo, the logistic chaos process is completely non-stationary or unstable. When more observations are generated, the mean and variance will continue to change. There is no convergence to a unique steady state equilibrium or to a few steady state equilibria. There are infinitely many!

Figure 15 is the same as Figure 14, but this time the dots representing the steady states are connected. This noise is a bit more persistent, i.e., moves a bit slower, than white noise:

Figure 14: Logistic Chaos (k=4; H=0.58); 1000 Obs.) is Not White Noise

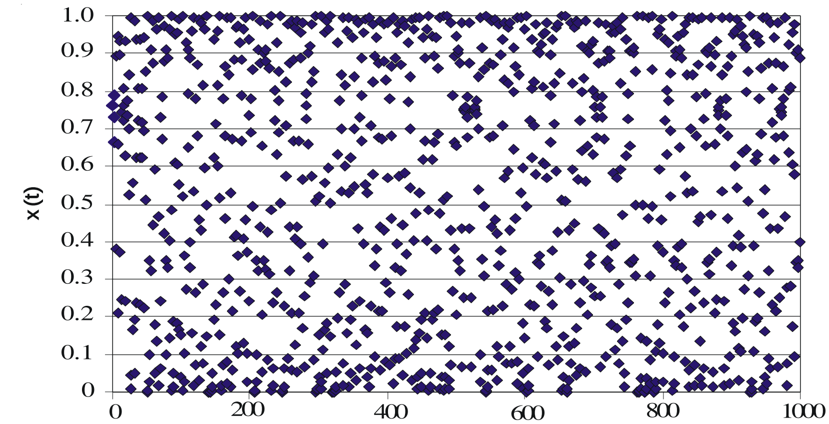


Figure 15: Logistic Chaos (k=4; H=0.58; 1000 Obs.)

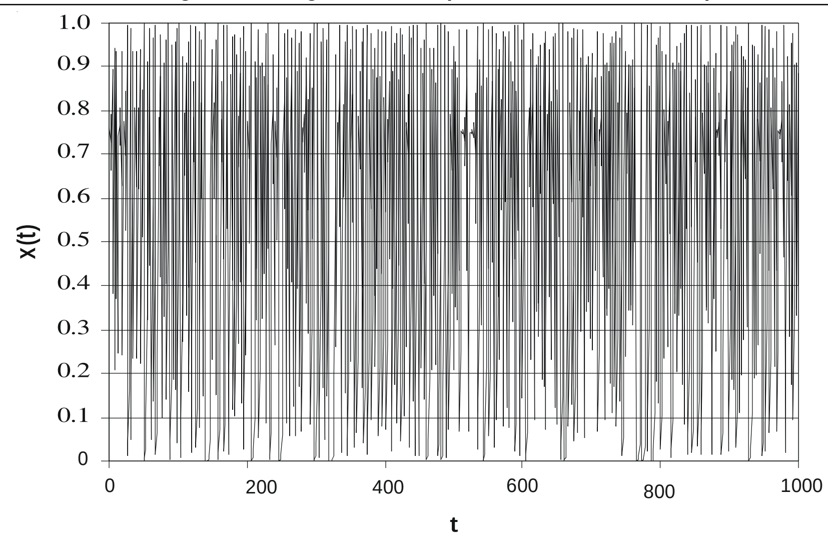
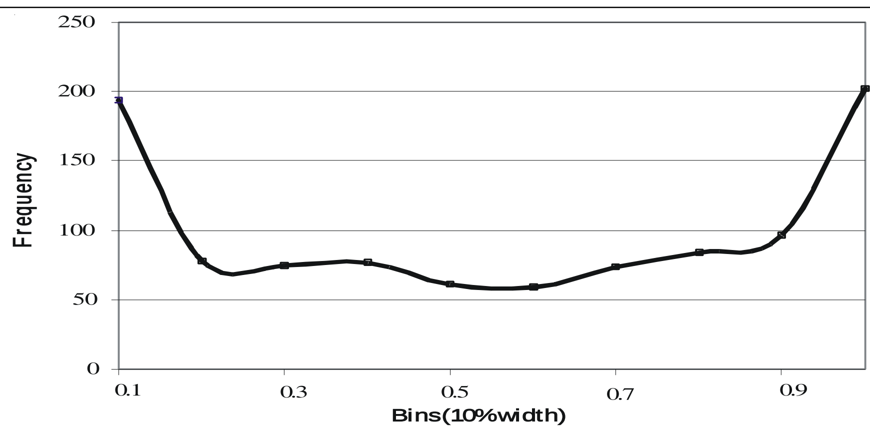


Figure 16: Frequency of x(t) Logistic Chaos (k=4; H=0.58; 1000 Obs.)



$0 < H = 0.58 > 0.5$ The fractal dimension of this continuous fractal (non-differentiable) space-filling line is: $D = 2 - H = 1.42$

Figure 16 shows the frequency distribution of chaos formed by computing a histogram with 10% equally spaced bins. This frequency distribution of the 1,000 values of the constrained $x(t)$, $0 < x(t) < 1$, is not flat, as would be the case with uniformly distributed white noise. It is highly platykurtic, with a kurtosis 04 = -1.48 (or normalized kurtosis = 3 - 1.48 = 1.52), compared with that of the Gaussian distribution's kurtosis 04 = 0 (or normalized kurtosis = 3). It has an imploded mode and very fat tails against $x = 0$ and $x = 1$ that are considerably heavier than the mode. It is an example of a stable distribution with a (Zolotarev) stability exponent: $D_Z \; \frac{1}{H} \; \frac{1}{0.583} = 1.715$. This very heavy tailed distribution jarringly contrasts with the conventional bell-shaped, unimodal, thin-tailed Gaussian distribution, with which most statisticians are familiar.

Nonlinear Dynamics

Many of the properties of the logistic parabola are paradigmatic, not only for other unimodal maps, but for different nonlinear maps as well. These maps model a broad range of contemporary problems in which nonlinearities play an essential role (Lyubitch, 2000). In order to better understand the concept of chaotic behavior of a financial system's evolutionary process, we must first generalize our definition of a dynamic system to include such nonlinear dynamic systems.

Definition 3: A dynamic system is described by its state at time t, which is a vector point $x(t)$ in a Euclidean phase space $R.^E$, with (integer) dimension E, and its evolution between time t and time $t + \Delta t$ is determined by certain invariant rules. Each point in phase space can be taken as the initial state x(0), and is followed by a trajectory x(t) for all $t > 0$.

Figure 17 shows the remarkable state space trajectory for the state vector $(x(t), x(t - 1))$ of the chaotic logistic process.

None of these trajectory cycles or orbits overlap (even under a microscope). How was this trajectory of 1,000 iterations, generated? Let's follow the first 10 iterations in Figures 18 and 19.

This is a clear example of Mandelbrot's non-periodic cyclicity (= orbits of different length). Figures 20 and 21 show the first 20 iterations. Followed by 50 iterations in Figures 22 and 23.

Figures 24 and 25 show the first 90 iterations. Notice how these points in state space lie precisely on a well-defined object, a parabolic curve, but the position of each these state points is completely irregular or unpredictable and no point is ever visited twice. Their positions depend on the precision of numerical computation of the logistic evolutionary trajectory, which depends on the length of the digital registers of the computer. A computer with a different computing precision, delivers a different series of points, as can be easily demonstrated. This is an example of deterministic, non-probabilistic randomness.

Fractal Attractor

What is the character of the chaotic logistic parabola as a process? To discuss this properly we need a new concept that is particular to nonlinear dynamic systems: the attractor.

Figure 17: State Trajectory of Chaotic Logistic Process (k=4)

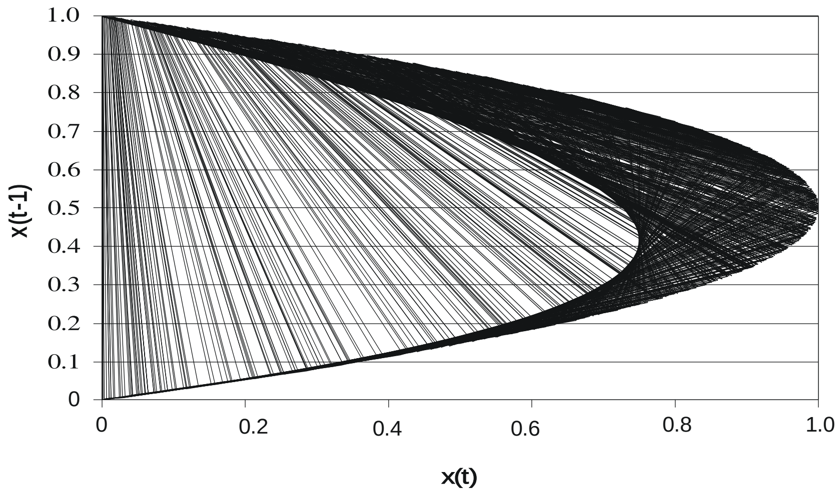


Figure 18

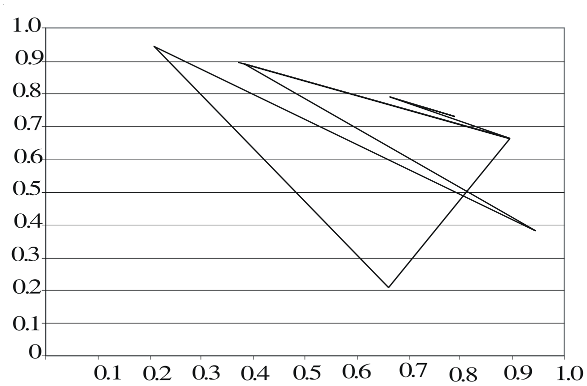


Figure 19

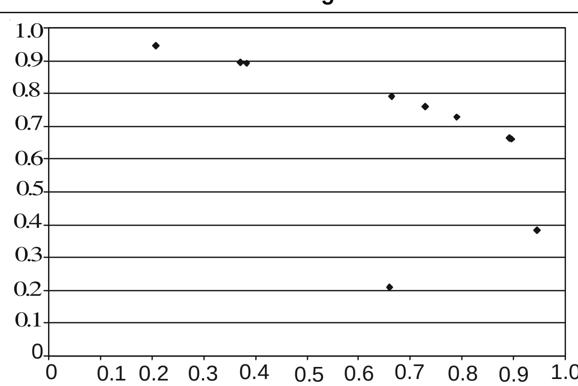


Figure 20

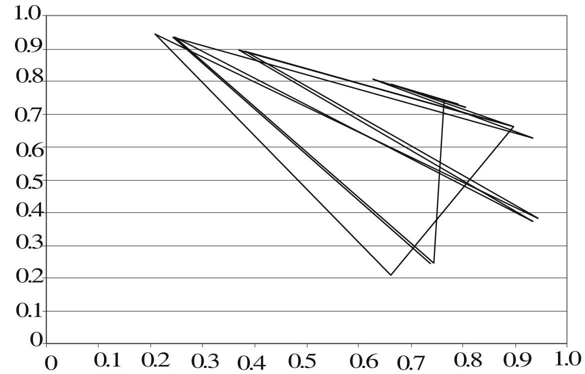


Figure 21

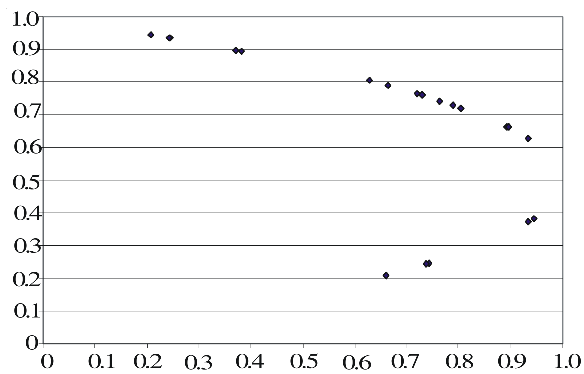


Figure 22

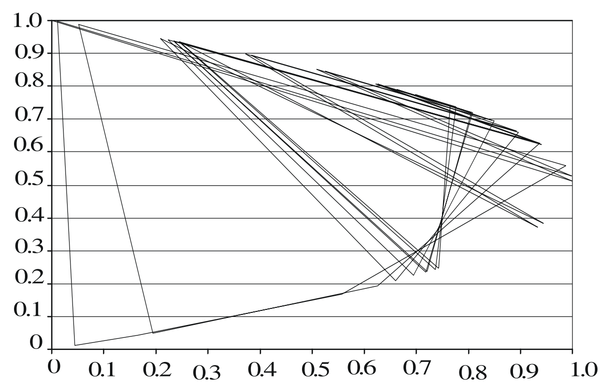


Figure 23

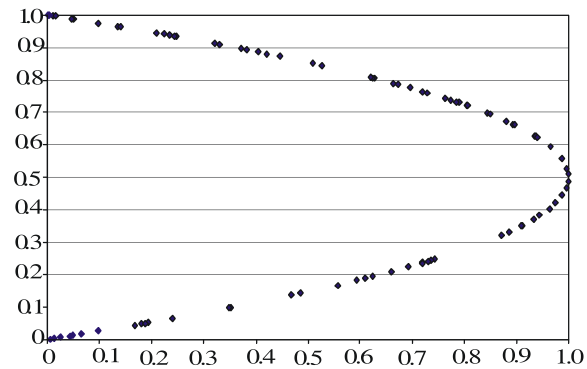


Figure 24

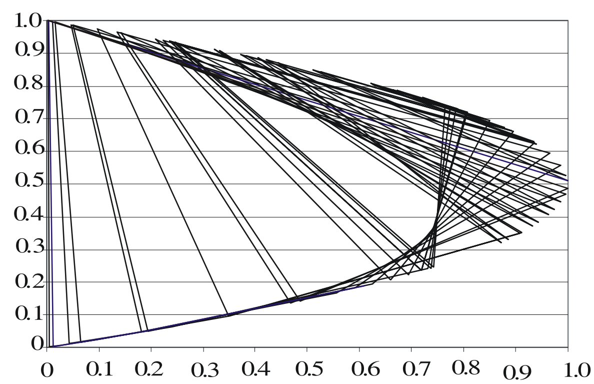
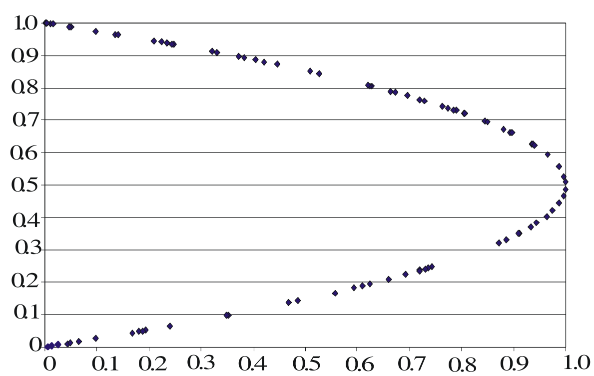


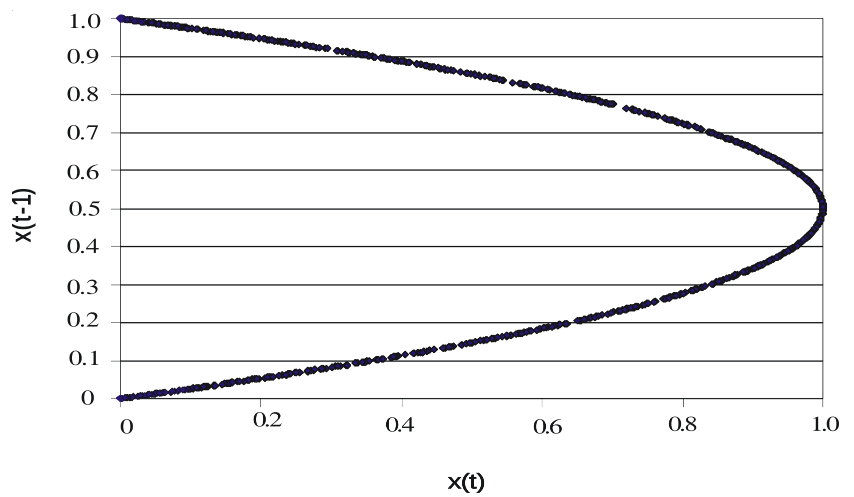
Figure 25



Definition 4: A dynamic process is said to have an attractor, if there exists a proper subset A of the phase space R^E, such that for almost all starting points x(0), and t large enough, x(t) is dose to some point of A.

In other words, the attractor set A is the subset of the Euclidean phase space with infinitely many equilibrium states x(t) of the system and their limiting points. The attractor characterizes the long-term behavior of the process. The well-defined parabolic object in state space in Figure 26, which defines the non-periodic cycles, is its attractor. Notice that this chaotic

Figure 26: Fractal Attractor of Chaotic Logistic Process (k=4)



process is not an anarchic process, since it clearly has macroscopic structure. It is complex in the sense that it combines the global stability of the (logistic) parabola with the local uncertainty of where the process at any time is on the parabola.

This subset of steady state points is a Cantor-like set, with a Haussdorff dimension close to (but not exactly equal to) zero. The attractor is the set of all these deterministically random equilibria of x(t). The limited dimension D of an attractor A is almost always fractional (= non-integer) and one speaks of fractal attractors ('also called strange or chaotic attractors). Fractal attractors are non-periodic, but cyclic! Their state trajectories in phase space never intersect, although these trajectories wander about the whole attractor set. Thus fractal attractors are sets with infinitely many dynamic equilibria.[14]

Remark 3: Fractal attractors are called strange, because familiar attractors have one of three distint forms. They consist either of single points (fixed or steady state points), finitely many points (periodic orbits), or continuous manifolds that give rise to periodic or aperiodic orbits. However, strange attractors do have structure and thus contain information, although this information is incomplete. Often they are self-similar or approximately so and they have fractal Hausdorff dimensions. Often we can identify the complete abstract set from the fractal attractor, as in the case of the logistic parabola.

[14] In classical financial-economics, equilibria are commonly static. Dynamaic equilibria have just started to appear in the financial-economic literature, although they were already familiar to mathematical economics in the 1970s.

Remark 4: The behavior of fractal attractors can be approximately described by wave functions, i.e., linear expansions of wavelets. This is currently a hot area of research: the modeling of (financial) turbulence by using wavelet multiresolution theory.

Chaotic Processes

We can now define chaotic behavior of a nonlinear dynamic process in terms of its attractor.

Definition 5: A chaotic process is one of which the behavior shows sensitivity to initial conditions x(0), so that no point of the attractor set A is visited twice in finite time.

Thus, any uncertainty in the initial state of the given system, no matter how small, will lead to rapidly growing errors in any effort to predict the future behavior of x(£). Meteorologists faces such problems with the prediction of the weather.[15] The plight of financial economists is, of course, similar (Los, 1991). Indeed, the transition from stable, equilibrium, behavior to chaotic behavior when the scaling parameter k is increased, as exhibited by the logistic parabola, has been observed in many physical systems, in fields as diverse as meteorology, seismology, ecology, epidemiology, medicine, economics and finance, to name just a few.[16] In particular, intermittency of turbulence, where some regions are marked by very high dissipation or chaos, while other regions seem by contrast nearly free of dissipation, is symptomatic of the observed behavior of FX markets, both in time and spatially. In the second half of 1997, the Southeast Asian markets saw a rapid succession of periods of turbulence and temporary stability. At the same time, while the Southeast Asian FX markets exhibited this temporal intermittency in the second half of 1997, the Japanese Yen and Deutschemark markets were completely unperturbed.

Conclusion

In this paper, we have studied the behavior of a nonlinear dynamic system by simulation in preparation of the quantitative study of FX market processes. The logistic parabola is capable of producing different process regimes, depending on the value of its scaling parameter k. These regimes are summarized in Table 2.

For the lower values of $k < 3$, the logistic process is like a stable linearized Newtonian process with a single stable equilibrium. Thus it is completely predictable in the short- and the long-term, or locally and globally. For $k = 4.0$ the logistic process is completely chaotic, i.e., it is unstable in both the short- and the long-term, or locally and globally unstable.

[15] Indeed, the modern study of chaotic dynamics began in 1963, when the American meteo-rologist Edward Lorenz demonstrated that a simple, deterministic model of thermal convection - in the Earth's atmosphere showed sensitivity to initial conditions or, in current terms, that it was a chaotic process.

[16] The term chaotic dynamics refers to the evolution of a process in time. Chaotic processes, however, also often display spatial disorder - for example, in complicated fluid flows. Incorpo-rating spatial patterns into theories of chaotic dynamics is now a very an active area of study. Researchers hope to extend theories of chaos to the realm of fully developed physical turbu-lence, where complete disorder exists in both space and time. This effort is widely viewed as among the greatest challenges of modern physics. The equivalence in financial economics would be to find a complete chaotic-dynamic theory of multiple coexisting market pricing processes, which can explain financial crises occurring in several interlinked regional pricing markets (Cf. Dechert, 1996).

Table 2: Equilibria Regime of the Logostic Process		
Parameter	**Periodic, Orbital Equilibria**	**Stability**
0 (K< 1	$x^* = 0$	Stable
k = 1	x;* =0	Marginally Stable
1<k <3	$x^* \frac{k \quad 1}{k}$	Superstable (1-period)
k = 3	$x^* = \frac{2}{3}$ o $x^* = \frac{2}{3}$ • (• very small) o $x^* = \frac{2}{3}$, etc	First Bifurcation (2-period)
3 < κ < 3.58	$x^*(l)$ o $x^*(2)$ o ... o $x^*(n+1) = x^*(l)$	Multiple Stability (n-period)
κ = 3.34		First 180 Degree Phase hift
κ = 3.45	$x^*(l)$ o $x^*(2)$ o $x^*(3)$ o $x^*(4)$ o $x^*(l)$,efc	Second bifurcation (4-period)
κ = 3.50		Second 180-degree phase shift
κ = 3.58	$x^*(l)$ o $x^*(2)$ o ... o x^*(large)	Moderate chaos
3.58 < κ < 4		Complexity
κ = 3.82	$x^*(l)$ o $x^*(2)$ o ... o x^*(large)	Moderate chaos
κ = 3.83	$x^*(l)$ o $x^*(2)$ o $x^*(3)$ o $x^*(l)$,efc	Apparent stability (3-period)
κ = 3.86	$x^*(l)$ o ... o $x^*(3)$ o $x^*(l)$ & $x^*(l)$ o ... o x^*(large)	Intermittency
κ = 4.0	$x^*(l)$ o $x^*(2)$ o ... o $x^*(f)$	complete chaos

A small change in the initial condition will cause it to move to a completely different level at an unpredictable time.

The most interesting process regimes from the point of view of current research into FX market processes, are the logistic processes that lie in between these two extreme regimes. These processes are complex and highly structured, like the period-doubling bifurcation processes, when $3 < k < 3.83$, which oscillate between even numbers of stable equilibria, i.e., with different but even period lengths. They can become very complex when the number of stable equilibria increases and they can show moderate chaos, but with intermittency. Periods of stability interlaced by periods of moderate chaos, or vice versa. Intermittency processes contain processes that are stable in the short-term or locally, and unstable in the long-term or globally, and processes that are unstable in the short-term, or locally, and stability in the long-term or globally.

Such complex processes are prevalent in nature because they survive. They always operate in high states of uncertainty. This is the same with FX markets. This uncertainty cannot be eliminated, because the buying and selling actions of the individual market participants cannot be predicted. But lack of local or short-term predictability may give such free pricing systems their global or long-term stability.

We saw that intermittency process regimes are very finely balanced. There are small ranges of the scaling parameter k, where the logistic process is in a period of calm and stability, but when the parameter moves outside these windows of stability ranges, the process is plunged into moderate chaos. This should provide cause for extreme caution for tinkering with well-working FX markets, which show intermittency, i.e., fairly long periods of relative stability, interrupted by fairly short periods of chaos. Institutional policy changes, which change parameter regimes, can cause a stable market mechanism to move into moderate chaos. On the other hand, it can also be rescued from such chaos, by counteracting policy changes. This does not mean that one should eliminate the uncertain pricing processes. But it does mean that we must first understand the actual quantitative parametrization of these processes before we start to tinker! It is clear that the current level of understanding of market pricing systems, which still relies on linearized models borrowed from Newtonian physics, is insufficient, because actual market pricing process are not both short- and long-term predictable. They show heterogeneous levels of predictability. The logistic parabola is just a simple analogue simulation model, but it provides some clear guidelines. However, more empirical research is required for the identification of the proper nonlinear dynamic configuration and parametrization of actual financial market pricing.

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Reference # 42J-2006-12-01-01

Bibliography

1. David A Patterson (1993), A New Approach to 1/0 Performance Evaluation Self-scaling 1/0 Benchmarks, Predicted 1/0 Performance, Proceedings of the 1993 ACM SIGMETRICS Conference on Measurment and Modeling of Computer Systems.

2. Dechert Davis (Ed.) (1996), *Chaos Theory in Economics: Methods, Models and Evidence*, (International Library of Critical Writings in Economics, No. 66), Edward Elgar Publisher, UK.

3. Feigenbaum F J (1979), "The Universal Metric Properties of Non-linear Transformations", *Journal of Statistical Physics*, Vol. 21, pp. 669-706.

4. Grassberger P (1981), "On the Hausdorff Dimension of Fractal Attractors", *Journal of Statistical Physics*, Vol. 26, pp. 173-179.

5. Grossman S and Thomae S (1977), "Invariant Distributions and Stationary Correlations of One-Dimensional Discrete Processes", *Zeitschrift fur Natur-forschung*, Vol. 32a, pp. 1353-1363.

6. Hall N (Ed.) (1991), *Exploring Chaos: A Guide to the New Science of Disorder*, W. W. Norton & Co., New York.

7. Hsieh A D (1991), "Chaos and Non-linear Dynamics: Applications to Financial Markets", *Journal of Finance*, Vol. 46.

8. Jensen R V and Myer C R (1985), "Images of the Critical Points of Nonlinear Maps", *Physics Review*, Vol. A32, pp. 1222-1224.

9. Karuppiah, Jeyanthi and Cornelis A Los (2000) *Wavelet Multiresolution Analysis of High-Frequency FX Rates, Summer 1997*, Economics Working Paper, No. 00-6, School of Economics, Adelaide University, September 20, 2000, 57 pages.

10. Li T Y and J A Yorke (1975), "Period Three Implies Chaos", *American Mathematical Monthly*, Vol. 82, pp. 985-992.

11. Lo Andrew W and Craig MacKinlay A (1999), "Long-Term Memory in Stock Market Prices", Chapter 6, in *A Non-Random Walk Down Wall Street*, Princeton University Press, Princeton, NJ, pp. 147-184.

12. Los Cornelis A (2000), *Computational Finance: A Scientific Perspective*, World Scientific Publishers, Singapore.

13. Los Cornelis A (1991), "A Scientific View of Economic Data Analysis", *Eastern Economic Journal*, Vol. 17, No. 1, January/March 1991, pp. 61-71.

14. Lyubich Mikhail (2000), "The Quadratic Family as a Qualitatively Solvable Model of Chaos", *Notices of the American Mathematical Society*, Vol. 47, No. 9, pp. 1042-1052.

15. Mandelbrot Benoit B (1963), "The Variation of Certain Speculative Prices", *The Journal of Business*, Vol. 36, pp. 394-419 and Vol. 45, 1972, pp. 542-543. Reprinted as Chapter E14, with Annotations and Corrections, in Fractals and Scaling in Finance: Discontinuity, Concentration, Risk, Springer Verlag, New York, 1997, pp. 371-418.

16. Mandelbrot Benoit B (1966), "Forecasts of Future Prices, Unbiased Markets and Martingale Models", *The Journal of Business*, Vol. 39, January (Special Supplement), pp. 242-255. Reprinted as Chapter E19, "Non-linear Forecasts, Rational Bubbles, and Martingales," with Annotations and Corrections, in *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk, Springer Verlag*, New York, 1997, pp. 471-491.

17. Muller U, Dacorogna M, Olsen R, Pictet O, Schwarz M and Morgenegg C (1990), "Statistical Study of Foreign Exchange Rates, Empirical Evidence of a Price Change Scaling Law, and Intra-Day Analysis," *The Journal of Banking and Finance*, Vol. 14, pp. 1189-1208.

18. Nicolis Gregoire and Ilya Prigogine (1998), *Exploring Complexity: An Introduction*, W H Freeman and Co., New York (Original 1989; 5th Print).

19. Peters Edgar E (1994), *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics*, John Wiley & Sons, Inc., New York.

20. Peters Edgar E (1999), *Patterns in the Dark: Understanding Risk and Financial Crisis with Complexity Theory*, John Wiley & Sons, Inc., New York.

21. Savit Robert (1988), "When Random is Not Random: An Introduction to Chaos in Market Prices", *Journal of Future Markets*, Vol. 8, p. 271.

22. Schroeder Manfred (1991), *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise*, W. H. Freeman and Co., New York.

23. Sharkovskii A N (1964), "Coexistence of Cycles of a Continuous Map of a Line into Itself", *Ukrainische Mathematische Zeitschrift*, Vol. 16, pp. 61-71.

24. Singer D (1978), "Stable Orbits and Bifurcations of Maps of the Interval", SI AM *Journal of Applied Mathematics*, Vol. 35, pp. 260-267.

25. Verhulst Pierre-Frangois (1845), "Recherches Mathematiques sur la Loi d'Accroissement de la Population", *Nouvelles Memoires de l'Academic Royale des Sciences et Belles-Lettres de Bruxelles*, Vol. XVIII, No. 8, pp. 1-38.