Discount Rates in Emerging Capital Markets

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The estimation of the discount rate for an investment project in conditions of risk relies upon two crucial assumptions: market completeness and well-diversified investors. Although, these two assumptions are tenable in developed capital markets, they are not suitable in emerging markets. In emerging markets, there are not enough twin securities to obtain a unique stochastic discount factor and therefore one project market value. Hence, investors usually face short selling and borrowing restrictions. Furthermore, these markets are plagued with non-diversified entrepreneurs that invest all their capital to undertake entrepreneurial adventures. In this research, one derives expressions for the project discount rate, using the fundamental pricing equation under incomplete capital markets in two extreme situations: when investors hold a well-diversified portfolio, and when they are not diversified at all. Although both situations may apply in developed and emerging capital markets, they apply especially to emerging markets. In fact, well-diversified investors, such as foreign mutual funds, increasingly invest in emerging markets, while the bulk of firms involves either small or medium enterprises owned by a single or a group of non-diversified entrepreneurs. The study concludes that although the Capital Asset Pricing Model (CAPM) cannot hold under incomplete markets, it is still a good approximation for well-diversified investors in emerging markets. At the same time, it is necessary to use a hurdle rate, based on the project total risk for the case of non-diversified entrepreneurs.

1. Introduction

One of the most important assumptions in the current financial theory of asset pricing is the existence of a complete market. A complete market arises when the number of non-redundant financial assets is equal to the number of subsequent states of nature, when there is a unique probability distribution underlying the price of financial assets or when the introduction of any new security into the capital market can be priced in a unique way. These three conditions are equivalent to each other because all require the availability of enough twin securities or

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dynamic portfolios to replicate the asset's risk in every state of nature at any future moment. Besides, there is another condition that yields a complete market: A complete market arises when investors can transfer freely their wealth across states of nature. In a complete market, investors may short-sell securities in order to transfer their wealth across states of nature and this condition is not equivalent to the previous ones. Whenever the former three conditions are not met, one says that the capital market is incomplete; if the fourth condition is not fulfilled, one says that the capital market is technically incomplete.

The concept of a complete market is crucial because it helps us to obtain a situation of equilibrium in the capital market through homogeneous expectations (e.g., the Capital Asset Pricing Model—CAPM) or through non-arbitrage (e.g., Arbitrage Pricing Theory—APT). However, the situation of non-arbitrage is less restrictive because it only requires the fourth condition. In fact, one may obtain an equilibrium situation even when the number of non-redundant securities is less than the number of subsequent states of nature, whenever investors may construct dynamic portfolios and trade without restrictions.

In general, the concept of a complete market is powerful one because it provides us with an equilibrium price for any financial asset that is unique and in this sense objective from the point of view of the market. This result is certainly appealing because regardless of any particular investor's believes and preferences, financial assets will be priced in a unique way.

Recently, different authors started highlighting the importance of incomplete markets[1]. This situation is important because it stresses the situation of disequilibria and also it may arise in developed and emerging economies. In a situation of disequilibria one cannot obtain a single and objective market value for an asset, either because twin securities are scarce, illiquid or because investors are facing trading restrictions.

Although, one may find occasionally these features in developed capital markets, surely you will find them in emerging capital markets[2]. Hence, the well-known models that one may use to valuate real investments cannot be applied directly in emerging capital markets. Of course, there are other reasons for this: Undesirable return distributions, scarce and unreliable historical data, partial integration of the emerging capital market, country risk, among others[3].

The traditional approach to value an investment project estimates the price the project would have if it were traded in the market. Since this value will be derived from the systematic risk of twin securities (betas) or dynamic portfolios in those markets, one needs to know either a full-fledged theory about how to price financial assets in emerging capital markets or the valuation consequences of adapting the valuation models.

In recent years several versions of equity valuation models have been offered, but none has successfully overcome all the above mentioned problems for its applicability in emerging markets, especially the problems related to the market incompleteness (few twin securities and trading restrictions) and the existence of non-diversified investors.

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1 See Hearings and Kluber (2000), Nau and McCardle (1991), Smith and Nau (1995), among others.

2 Bris et al. (2003) have shown significant trading restrictions in emerging markets, especially related to short selling and borrowing restrictions.

3 Mongrut (2004a) explains several problems that one must face in order to valuate real investment projects in the emerging markets of South America.

In fact, most of the proposals have been directly towards defining new ways of estimating the parameters of the Capital Asset Pricing Model (CAPM), specifically how to estimate the risk-free rate, betas and market risk premiums in emerging markets, taking into account country risk. For example, Damodaran (1999a and 1999b) has made two proposals concerning the estimation of the risk-free rate in emerging markets and the emerging market risk premium, taking into account country risk. Pereiro and Galli (2000) and Pereiro (2001) proposed a method to valuate closely held firms in emerging markets, taking into account country risk and other factors such as ownership concentration, liquidity and size effects.

Concerning systematic risk, Godfrey and Espinoza (1996) and Lessard (1996) have made proposals concerning the estimation of offshore project betas in emerging markets. In particular, Lessard suggested that betas from well-developed countries could be translated into betas for emerging markets. However, as Bodnar et al. (2003) have shown, this proposal is flawed for emerging markets.

Another string of research tried to incorporate the fact that investors in emerging markets are facing a significant downside risk. In this sense, Estrada (2000 and 2002) has made a proposal concerning the estimation of downside risk betas. However, this approach does not account for partial integration and for capital market incompleteness.

It is a well-known fact that the process of liberalization of emerging capital markets at the beginning of the 1990s brought down the cost of equity capital (Stulz 1999), so it makes sense to consider the degree of capital market integration in the estimation of the project discount rate. In this sense, Bekaert and Harvey (1995) tried to incorporate the fact that the degree of integration varies through time by mixing the local CAPM with a global version of the CAPM. Unfortunately, their model is hard to apply and it only considers the case of well-diversified investors.

Other proposals are related to the estimation of hurdle rates for emerging markets as a whole and the use of the Decision Analysis Approach (DAA) and Contingent Claim Valuation (CCV) to assess real investments and their attached real options in incomplete markets. Specifically, Erb et al (1996) have proposed a Country Credit Rating (CCR) model for the estimation of hurdle rates for emerging markets with and without capital markets, while Smith and Nau (1995) have proposed the use of Decision Analysis for the valuation of real options in incomplete markets.

Unfortunately, the CCR model cannot be applied to single investment projects. The Decision Analysis view uses the single-agent optimality approach to accomplish a subjective valuation of the project. In this approach, one specifies the beliefs (i.e., return probability distribution) and preferences (i.e., utility function) of incumbent investors and experts, instead of assuming that investors have homogeneous expectations (equilibrium approach) or that they are risk neutral (non-arbitrage approach).

Although the CAPM cannot hold under incomplete markets, it performs well in this situation. In this respect, Hearings and Kluber (2000) have shown that the CAPM provides a good benchmark for equilibrium prices in a two period general equilibrium model with incomplete markets even when agents are not mean-variance optimizers and returns are not normally distributed.

They approximate equilibrium for various types of investors' preferences and beliefs and found that although the CAPM cannot hold exactly for the chosen specification, the pricing-errors were extremely small. This result provides the basis for using the CAPM in incomplete markets. Indeed, if pricing-errors are extremely small, why not using a quadratic utility function (that yields the CAPM) albeit it assumes increasing absolute risk aversion? In this sense, it makes more sense to use this utility function simply because the cost of much more realism does not pay off for the case of a well-diversified investor. The case of a non-diversified investor is more difficult to make because there is no reason to believe that pricing-errors with a hurdle rate are also extremely small. So, there is more degree of freedom and the results will not longer be objective.

In contrast to the previous literature, this research aims to unveil the implications of the emerging market incompleteness and the high proportion of non-diversified investors for the estimation of the project's discount rate. One is referring to project discount rate instead of the project cost of equity, because the later is a special case of the former when the only relevant risk is the systematic one. As one would see, the discount rate could also be a hurdle rate based upon the project total risk. The selection of the appropriate discount rate will depend upon the marginal investor's degree of diversification.[4]

Although, corporate managers could be able to estimate utility functions (each one would imply a different pricing model), it is unlikely that small and medium entrepreneurs will perform such tasks. Pastor and Stambaugh (1998) have shown that uncertainty about which pricing model to use is less important (on average) than within-model parameter uncertainty. Since risk aversion is bounded for human beings, it would be better to derive a set of expressions for the project's discount rate with a particular utility function and according to empirical extreme values for the risk aversion coefficient. This is particularly valuable in the case of non-diversified entrepreneurs, because they are usually left without a mathematical expression to estimate the project's hurdle rate.

In this research one attempts to answer the following questions: How does the emerging market incompleteness influence the estimation of the project discount rate for well-diversified investors? What set of expressions could non-diversified entrepreneurs use in order to estimate the project's hurdle rate?

In particular, one uses the fundamental pricing equation under two alternative approaches for valuation: The equilibrium approach and the single-agent optimality approach. The first approach lets us derive the CAPM and analyze its consequences under incomplete markets for well-diversified investors, while the second approach helps us to derive a hurdle rate consistent with the project total risk that suits well the situation of non-diversified investors in emerging markets[5]. For the former case a range is obtained, so the cost of equity capital may fluctuate within the boundaries and so the project added value. In the case of the project's hurdle rate, only a lower bound can be obtained that conforms to the highest project's added value given the entrepreneur's risk aversion and reward-to-variability ratio.

[4] For traded firms, the marginal investor is the one that, with the highest probability, will buy the firm's shares in the next trading day.

[5] One considers uncertainty as bounded uncertainty or risk.

The paper is organized as follows: Section 2 provides an introduction to the fundamental pricing equation. Section 3 is devoted to discuss briefly the concept of incomplete market, while Section 4 restates the fundamental pricing equation under incomplete capital markets. In Sections 5 and 6, one derives the boundaries for the project discount rate under the assumption of well-diversified investors in incomplete markets and under the assumption of non-diversified entrepreneurs in incomplete markets, respectively. Section 7 concludes the work.

2. The Fundamental Pricing Equation

The fundamental pricing equation, albeit with a different name, has been obtained originally in the work of Arrow (1963), Debreu (1959), and Hirshleifer (1964) in the context of general equilibrium. Afterwards, Myers (1968) generalized it into the context of an incomplete market by including the effects of trading restrictions such as short-selling and borrowing restrictions. Finally, Cochrane (2001) has restated the whole asset pricing theory, using this framework. In this section, one derives the fundamental pricing equation following closely Campbell (2000) and Cochrane (2001).

Consider a rational investor 'i' that must decide how much to consume today (period 't') and tomorrow (period 't+1'). The preferences for this investor are captured by the following utility function:

$$U_{i,w}\left\{C_t, \sum_{w \in \Omega} \pi(w) U_{i,w}(C_{t+1}(w))\right\} = U_{i,w}\left(C_t, E_t(U_{i,w}(C_{t+1}(w)))\right)$$

One may see that the investor's utility function depends on the prevalent state of nature 'w' in 't+1', where 'Ω' is the set of possible states of nature for tomorrow. Furthermore, the part of the utility function for tomorrow is conditioned on the information we have today—so it is a conditional expectation. Note also that in this conditional expectation one uses the subjective probability of the investor for each state of nature.

Now, assume that the utility function is increasing, concave and continuous everywhere. The first assumption represents the fact that more is better than less; the second refers to the decreasing utility of one additional unit; and the third guarantees an optimum. Besides, assume that the utility function is time and state-separable (durability and habit persistence) and that the constant 'β' represents the investor's impatience, so it is a subjective discount factor.

With these assumptions, one can rewrite the investor's utility function as follows[6]:

$$U_{i,w}\left(C_t, E_t(U_{i,w}(C_{t+1}(w)))\right) = U_i(C_t) + \beta E_t U_i(C_{t+1}(w))$$

The objective for the agent or investor is to maximize this utility function subject to some constraints related to his consumption today and tomorrow. Assume that there are no trading restrictions, so the investor may short-sell financial assets or borrow at the risk-free rate.

[6] These are standard assumptions in the literature.

Now, assume that there are 'n' financial assets in the market where the price of the 'j' financial asset today is given by '$p_{t,j}$'. This financial asset will generate a cash flow tomorrow '$F_{t+1,j}(w)$' that will depend on the prevalent state of nature[7]. Furthermore, assume that his original consumption level (if he does not buy any financial asset) is denoted by 'e_t'. Given these assumptions, the investor wants to know how many titles 'ξ_j' of the 'j' financial asset he must buy or sell today in order to maximize his utility function.

One can write the investor's problem in the following way:

$$\mathrm{Max}_{\{\xi_j,\, j=1,2,\ldots,n\}}\, U_i(C_t) + \beta E_t\left[U_i(C_{t+1}(w))\right]$$

Subject to:

$$C_t = e_t - \sum_{j=1}^{n} \xi_j p_{t,j}$$

$$C_{t+1}(w) = e_{t+1} + \sum_{j=1}^{n} \xi_j F_{t+1,j}(w)$$

Replacing the two restrictions in the objective function and deriving the objective function with respect to 'ξ_j', one obtains 'n' first-order conditions, one for each financial asset. In the case of the 'j' financial asset, one has:

$$E_t\left[\beta U_i'(C_{t+1}(w))F_{t+1,j}(w)\right] = U_i'(C_t)P_{i,j} \quad j = 1,2,\ldots,n \qquad \cdots\text{(1a)}$$

The left-hand side of equation 1 a represents the marginal benefit (increase in utility) that the investor obtains from the additional payoff tomorrow. The right-hand side of the equation reflects the loss in utility if the investor buys one additional unit of the financial asset. This condition simply states that the investor will continue to buy or sell the financial asset until the marginal gain equals the marginal loss.

This equation can be stated in different ways. The most popular expression for the fundamental pricing equation is the following:

$$E_t\left[\beta \frac{U_i'\left(C_{t+1}(w)\right)}{U_i'(C_t)} F_{t+1,j}(w)\right] = E_t\left[m_{t+1}(w)F_{t+1,j}(w)\right] = P_{t,j} \qquad \cdots\text{(1b)}$$

Where:

$$m_{t+1}(w) = \beta \frac{U_i'\left(C_{t+1}(w)\right)}{U_i'(C_t)} \text{ is the stochastic discount factor (SDF)}$$

Alternatively, if one defines the conditional expectation, one obtains:

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[7] If the financial asset is one stock, it will yield a dividend tomorrow.

$$\sum_{w} \pi(w) \frac{U_i'(C_{t+1}(w)) F_{t+1,j}(w)}{U_i'(C_t)} = P_{t,j} \qquad \cdots(1c)$$

If one divides both sides of equation 1b by the price of the financial asset today, one obtains the fundamental pricing equation in gross returns[8] :

$$1 = E_t\left[m_{t+1}(w) R_{t+1,j}(w)\right] \qquad \cdots(1d)$$

Where:

$$R_{t+1,j}(w) = \frac{F_{t+1,j}(w)}{P_{t,j}}$$

Finally, using net returns instead of gross returns[9]:

$$1 = E_t\left[m_{t+1}(w) * (1 + r_{t+1,j}(w))\right] \qquad \cdots(1e)$$

Where:

$$R_{t+1,j}(w) = 1 + r_{t+1,j}(w)$$

All these expressions (la-le) are just alternatives, but equivalent ways to express the fundamental pricing equation in discrete time[10].

Equation 1d tells us that although the returns for the 'j' financial asset may vary tomorrow across states of nature, the conditional expected present value of the returns must always be equal to one. Furthermore, the expression '$m_{t+1}(w)$' is called Stochastic Discount Factor (SDF) and let us interpret the fundamental pricing equation as the state-price-weighted average of the payoffs (cash flows) in each state of nature[11]. Given that there are no arbitrage opportunities, there will be a positive set of state prices and there will be a positive SDF.

It is important to note that in the derivation of the fundamental pricing equation there is no assumption that markets are complete, the existence of a representative agent, that asset returns or cash flows are normally distributed, that investors have no human capital or labor income, a quadratic utility function, or that the market is already in equilibrium. As Cochrane (2001) points out, all these assumptions come later as special and convenient cases for asset pricing. However, what one has assumed is that investors can freely buy and sell assets in the amounts they want and that investors maximize their utility function in a two-period setting, although this two-period setting can be replicated for any two periods in an intertemporal setting.

8 Gross returns includes the capital as opposed to net returns, which do not include the capital.
9 This expression is useful for empirical purposes.
10 Cochrane (2001) shows the continuous time versions of some of the expressions reported here.
11 The stochastic discount factor is also known as the marginal rate of substitution, the pricing kernel or the state-price density.

There are two interesting cases that can be derived from the fundamental pricing equation: The pricing of a risk-free asset and the pricing of a risky asset. In the former case, from equation 1d, we obtain:

$$1 = E_t\left[m_{t+1}(w)R_{t+1,j}(w)\right] = E_t\left[m_{t+1}R_f\right] \Rightarrow R_f = \frac{1}{E_t\left[m_{t+1}\right]} \qquad \cdots(2a)$$

This result shows that in general the return of a risk-free asset is equal to the inverse of the conditional expectation of its stochastic discount factor. Of course, whenever the risk-free rate is constant, the SDF is also constant. The second case corresponds to the pricing of a risky asset. One again uses expression 1d and defines the covariance between the SDF and the gross return of the risky asset:

$$1 = E_t\left[m_{t+1}(w)R_{t+1,j}(w)\right] = E_t(m_{t+1}(w))E_t\left[R_{t+1,j}(w)\right] + Cov(m_{t+1}(w), R_{t+1,j}(w)) \qquad \cdots(2b)$$

If one inserts expression 2a into the previous one and solve for the conditional expectation of the risky asset, one obtains:

$$E_t(R_{t+1,j}(w)) = R^f + \left[-\frac{Cov(m_{t+1}(w), R_{t+1,j}(w))}{E_t(m_{t+1}(w))}\right] \qquad \cdots(3)$$

One can write this expression in the following way:

$$E_t(R_{t+1,j}(w)) = R^f + \left[\frac{Cov(m_{t+1}(w), R_{t+1,j}(w))}{Var(m_{t+1}(w))}\right]\left[-\frac{Var(m_{t+1}(w))}{E_t(m_{t+1}(w))}\right] = R_f + \beta_{t+1,j}\theta_{t+1} \qquad \cdots(4)$$

Formula 3 shows that the conditional expectation of the return of a risky asset depends, in general terms, on the covariance between the SDF and the risky asset return in every state of nature and the conditional expectation of the SDF. Equation 4 shows the beta-pricing model. It states that each conditional expected return should be proportional to beta. Note that beta varies across financial assets, while the price of risk (theta) is the same for all assets. This beta is really forward-looking and in this sense historical data does not help us much to estimate it unless there is reliable data and betas are stationary through time. If this is the case, beta could be estimated using a traditional regression analysis where the dependent variable is the return of the financial asset and the independent variable is the discount factor 'm' (Cochrane, 2001). Finally, if one replaces the expression of the SDF in formula 3, one obtains the following:

$$E_t(R_{t+1,j}(w)) = R^f + \left[-\frac{Cov(U_i(C_{t+1}(w)), R_{t+1,j}(w))}{E(U_i(C_{t+1}(w)))}\right] \qquad \cdots(5)$$

This expression tells us that any asset that moves together positively with consumption makes consumption more volatile, so it must offer a higher return in order to hold it.

3. Complete and Incomplete Capital Markets

The concept of an incomplete capital market is central to this research, so this section is devoted to specify the concept formally. An incomplete market arises once one relaxes one of the conditions needed to have a complete market. The following conditions must be met in order to have a complete market (Giménez, 2001; Duffie, 1996; and Mas-Collel et al, 1995):

- The number of non-redundant financial assets must be equal to the number of subsequent states of nature.
- There must be a unique probability distribution underlying the prices of the financial assets. In other words, only one state price process verifies the fundamental pricing equation.
- The introduction of any new financial asset into the capital market can be priced in a unique way.
- Investors may transfer their wealth across states without restrictions.

A way to understand these conditions is by using the fundamental pricing equation from the previous section. Using a bit of linear algebra, one may express formula 1c in the following way:

$$\begin{bmatrix} F_{t+1,1}(w_1) & \cdots & F_{t+1,1}(w_N) \\ F_{t+1,2}(w_1) & \cdots & F_{t+1,2}(w_N) \\ \vdots & & \vdots \\ F_{t+1,N}(w_1) & \cdots & F_{t+1,N}(w_N) \end{bmatrix}_{N \times N} \begin{bmatrix} \pi(w_1)m_{t+1}(w_1) \\ \pi(w_2)m_{t+1}(w_2) \\ \vdots \\ \pi(w_N)m_{t+1}(w_N) \end{bmatrix}_{N \times 1} = \begin{bmatrix} P_{t,1} \\ P_{t,2} \\ \vdots \\ P_{t,N} \end{bmatrix}_{N \times 1} \qquad \cdots(6a)$$

$$F \times m = p$$

Matrix "F" contains the dividends that the "N" financial assets will pay in the "N" subsequent states of nature "W". Vector m contains the product of the investor's preference (implicit in the stochastic discount factor) and believes (subjective probabilities—π). Finally, vector p contains current prices of the "N" financial assets.

The first condition for a complete market states that the number of non-redundant securities must equal the number of subsequent states of nature. According to formula 6, this amounts to say "N" financial assets and "N" subsequent states of nature. In other words, the columns or rows of matrix F must be linearly independent[12]. Given this, the rank of the squared matrix F must be "N", and it must be invertible:

$$m = F^{-1}p \qquad \cdots(6b)$$

[12] The "N" dividends for each state of nature are linearly independent if none column vector or row vector can be represented as a linear combination of the other "N-l" column or row vectors.

Given the first condition, equation 6b must have a unique solution (vector m**)**. This vector describes the probability distribution of the stochastic discount factor (state-price density) where each product represents a contingent claim, which depends on the realization of a particular state of nature.

A contingent claim or state-price is the present value of one monetary unit (in this case consumption unit) that financial asset "s" will pay only if state of nature "Ws" occurs tomorrow, while the remaining assets will not pay off. If the state of nature "Ws" does not occur tomorrow, there will be another financial that will payoff.

Conditions B and C for a complete market are consequences of condition A. Indeed, if equation 6b has only one solution, there must be only one state-price density for the stochastic discount factor and financial assets will have only one price. Furthermore, any new financial asset will be redundant because there is already enough number of financial assets; so, the new financial asset will have a unique price too.

Condition D is quite different from the other three. If there is enough number of financial assets and the investor does not face any restriction to trade with them, he can transfer freely his wealth or consumption across states. If the investor cannot short sell securities or if he cannot borrow at the risk-free rate, he will not be capable to transfer his wealth across states of nature. In this sense, condition A is necessary, but no sufficient condition to have a complete market.

As Giménez (2001) has pointed out, whenever investors cannot trade freely with financial assets, the capital market will be technically incomplete. Assuming that this situation affects all investors in the market, there will be shadow prices for the financial assets, so they will not longer have a unique price. If condition D is not met for few investors in the market, then condition A will be necessary and sufficient for the majority of investors that can trade freely in the market.

In brief, an incomplete market is a situation that arises whenever conditions A or D are not met. In this case, every financial asset will not longer have a unique price.

4. The Fundamental Pricing Equation in Incomplete Capital Markets

In this section, one translates the work of Myers (1968) using the stochastic discount factor. Specifically, one derives the fundamental pricing equation giving a short selling restriction. The problem to solve is very similar to the previous one, but in this case one needs to add new restrictions:

$$\text{Max}_{\{\theta_j\},\, j=1,2,\ldots,n}\; U_i(C_t) + \beta E_t\left[U_i(C_{t+1}(w))\right]$$

Subject to:

$$C_t = e_t - \sum_{j=1}^{n} \theta_j p_{t,j}$$

$$C_{t+1}(w) = e_{t+1} + \sum_{j=1}^{n} \alpha_j F_{t+1,j}(w)$$

$$\alpha_j \geq 0$$

The crucial restriction here is the fact that short selling is not permitted ($\alpha_j \geq 0$), so the problem is a non-linear one that must be solved using the Kuhn-Tucker conditions. The Lagrangean is the following:

$$L = U_i(C_t) + \delta_t U_i(C_{t+1}(w)) + \lambda_t \left[e_t - \sum_{j=1}^{n} \alpha_j P_{t,j} - C_t \right]$$

$$+ \gamma_t \left[e_{t+1} + \sum_{j=1}^{n} \alpha_j F_{t+1,j}(w) - C_{t+1}(w) \right] \qquad \cdots(7a)$$

Where " λ_t" and " γ_t" are the marginal effect in utility, the Lagrangean multipliers. The following Kuhn-Tucker conditions must be satisfied:

$$\begin{array}{lll}
\dfrac{\partial L}{\partial C_t} \leq 0 & C_t \geq 0 & \dfrac{\partial L}{\partial C_t} C_t = 0 \\
\dfrac{\partial L}{\partial C_{t+1}} \leq 0 & C_{t+1}(w) \geq 0 & \dfrac{\partial L}{\partial C_{t+1}(w)} C_{t+1}(w) = 0 \\
\dfrac{\partial L}{\partial \lambda_t} \geq 0 & \lambda_t \geq 0 & \dfrac{\partial L}{\partial \lambda_t} \lambda_t = 0 \\
\dfrac{\partial L}{\partial \gamma_t} \geq 0 & \gamma_t \geq 0 & \dfrac{\partial L}{\partial \gamma_t} \gamma_t = 0 \\
\dfrac{\partial L}{\partial \alpha_j} \leq 0 & \alpha_j \geq 0 & \dfrac{\partial L}{\partial \alpha_j} \alpha_j = 0
\end{array} \qquad \cdots(7b)$$

An investor must assign his wealth between consumption and investment, but in such a way that he consumes something, as a consequence:

$$C_t > 0 \qquad C_{t+1} > 0 \qquad \cdots(7c)$$

Then:

$$\frac{\partial L}{\partial C_t} = 0 \quad \frac{\partial L}{\partial C_{t+1}} = 0 \qquad \cdots(7d)$$

Given this observation, the Kuhn-Tucker conditions are the following:

$$\frac{\partial L}{\partial C_t} = U'_i(C_t) - \lambda_t = 0 \qquad \cdots(7e)$$

$$\frac{\partial L}{\partial C_{t+1}(w)} = \delta E_t[U'_i(C_{t+1}(w))] - \gamma_t = 0 \qquad \cdots(7f)$$

$$\frac{\partial L}{\partial \lambda_t} = e_t - \sum_{j=1}^{n} \alpha_j P_{t,j} - C_t \geq 0 \qquad \cdots(7g)$$

$$\frac{\partial L}{\partial \gamma_t} = e_{t+1} + \sum_{j=1}^{n} \alpha_j F_{t+1,j}(w) - C_{t+1} \geq 0 \qquad \cdots(7h)$$

$$\frac{\partial L}{\partial \alpha_j} = -\lambda_t [P_{t,j}] + \gamma_t [F_{t+1,j}(w)] \leq 0 \qquad \cdots(7i)$$

Dividing (7e) by (7f) one gets:

$$\frac{\lambda_t}{\gamma_t} = \frac{U'_i(C_t)}{\delta E_t[U'_i(C_{t+1}(w))]} \qquad \cdots(7j)$$

From equation (7i) one has:

$$\frac{\lambda_t}{\gamma_t} \geq \frac{F_{t+1,j}(w)}{P_{t,j}} \qquad \cdots(7k)$$

Equation (7j) together with equation (7k) yields the lower limit of the security price:

$$P_{t,j} \geq E_t\left[\delta \frac{U'_i(C_{t+1}(w))}{U'_i(C_t)} F_{t+1,j}\right] \Rightarrow 1 \geq E_t[m_{t+1} R_{t+1,j}] \qquad \cdots(8a)$$

Hence, the lower limit is associated to the case of a technically incomplete capital market where investors face short selling restrictions. In order to derive the upper limit, one needs to assume that there are two types of investors in the capital market: investors for which the short selling restriction is bidding and investors for which the short selling is not bidding. According to Myers (1968) short selling is an operation that may be regarding as purchasing a "dummy" security with a vector of contingent returns $R^*_{t+1,j}$ that are derived from the original asset's returns $R_{t+1,j}$.

In the absence of margin requirements to cover unfavorable events during the short selling operation, the returns of this dummy security are the mirror image of the returns from the original security provided that the "dummy" security is held until the end of the investor's investment horizon. Hence, investors that are able to short sell securities may replace in equation "7i" the following identities:

$$P^{*}_{t\ 1,j} \qquad P_{t\ 1,j}$$

$$R^{*}_{t\ 1,j} \qquad R_{t\ 1,j}$$

Hence, one obtains the upper limit of the security price:

$$P_{t,j} \quad dE_t \; {}^{a}_{\neg}« \frac{U_i'(C_{t\ 1}(w))}{U_i'(C_t)} F_{t\ 1,j} \; {}^{o}_{¼}» \ddot{Y}\, 1 \; dE_t \; \grave{m}_{t\ 1} R_{t\ 1,j} \; @ \qquad \cdots(8b)$$

However, equilibrium prices do not arise from equations 8a and 8b, because some investors cannot short sell some securities. In other words, every investor in the market cannot hold at least marginal amounts of each security, either short or long in his portfolio. In this situation, there will be a gap between equations 8a and 8b, they are simply inconsistent to each other. In this capital market where some investors may short sell securities and others not, there will be a lower limit and an upper limit for the price of each security.

5. The Cost of Equity for Well-diversified Investors in Incomplete Markets

Myers (1968) devised a full-fledged time-state-preference model for security valuation. His model was in fact a generalization, at that time, of the fundamental pricing equation.

In this section, one extends the work of Myers by deriving the CAPM in incomplete capital markets. One shall see that in a market where some investors can short sell and some cannot, it is possible to obtain bounds for the cost of equity capital in the case of well-diversified investors.

In order to derive the CAPM, one uses the following power utility function for time "t" with constant preferences (Huang and Litzenberger 1988):

$$U(C_t) \quad \frac{1}{B\ \ 1}(C_t \quad BC^{*})^{1\ \frac{1}{B}} \qquad t \qquad \cdots(9a)$$

In the above expression C* is the Pareto optimal consumption allocation for an individual with certain weightings. This power utility function is for a representative individual so one can aggregate this function for the market. "B" is assumed to be constant across individuals and one also assumes that the individuals' utility functions are increasing and strictly concave. Following Cochrane (2001), suppose there exists a representative agent whose utility function for time t+1 consumption is a function with the same B and equal to -1, then the intertemporal utility function is:

$$U(C_t C_{t\ 1}) \qquad \frac{1}{2}(C_t \quad C^{*})^2 \quad \frac{1}{2} \,/\, (_t\ (C_{t\ 1} \quad C^{*})^2 \qquad \cdots(9b)$$

The representative individual should maximize his utility function subject to the following budget constraints:

$$C_{t+1} = W_{t+1}$$

$$W_{t+1} = R^{W}_{t+1}(W_t - C_t)$$

$$R^{W}_{t+1} = \sum_{i=1}^{N} W_j R_{t+1,j} \qquad \cdots(9c)$$

$$\sum_{i=1}^{N} W_j = 1$$

Where:

- W_t : Is the total wealth in time t
- W_{t+1} : Is the total wealth in time t+1
- R^{W}_{t+1} : Is the rate of return on a claim to total wealth in time t+1
- W_j : Is the weight of security i in the portfolio
- $R_{t+1,j}$: Is the rate of return of security i

The stochastic discount factor is:

$$m_{t+1} = \beta \frac{\frac{\partial U}{\partial C_{t+1}}}{\frac{\partial U}{\partial C_t}} = \beta \frac{(C_{t+1} - C^*)}{(C_t - C^*)} \qquad \cdots(10a)$$

Introducing the first two constraints (those associated to a two period problem), one obtains a more precise expression for the stochastic discount factor:

$$m_{t+1} = -\frac{\beta C^*}{C_t - C^*} + \frac{\beta(W_t - C_t)}{C_t - C^*} R^{W}_{t+1} = a_t + b_t R^{W}_{t+1} \qquad \cdots(10b)$$

$$a_t = -\frac{\beta C^*}{C_t - C^*}, \quad b_t = \beta \frac{(W_t - C_t)}{(C_t - C^*)}$$

From expression 10b one concludes that the stochastic discount factor is linear and that it involves a total wealth portfolio return R^{W}_{t+1}. The only two parameters that one needs to know are "a" and "b". In order to obtain them, one must assume that the CAPM prices correctly any two securities. A convenient choice is the security that yields a risk free return and the market portfolio that may proxy for the total wealth portfolio:

$$1 = E_t\left[m_{t+1}\right]R_f \qquad \cdots(11a)$$

$$1 = E_t\left[m_{t+1}R^W_{t+1}\right] \qquad \cdots(11b)$$

Now plugging the stochastic discount factor 10b into the above expressions yields[13]:

$$1 = \left(a_t + b_t E_t\left[R^W_{t+1}\right]\right) R_f \Rightarrow a_t = \frac{1}{R_f} - b_t E_t\left[R^W_{t+1}\right] \qquad \cdots(11c)$$

$$1 = \left(a_t + b_t E_t\left[R^W_{t+1}\right]\right) E_t\left[R^W_{t+1}\right] + b_t Var\left[R^W_{t+1}\right] \qquad \cdots(11d)$$

Replacing expression 11c into expression 11d yields the following expression:

$$1 = \left[\frac{1}{R_f}\right] E_t\left[R^W_{t+1}\right] + b_t Var\left[R^W_{t+1}\right]$$

Solving this expression for "b_t" yields:

$$b_t = -\frac{E_t\left[R^W_{t+1}\right] - R_f}{R_f Var\left[R^W_{t+1}\right]} \qquad \cdots(11e)$$

In the second section, one stated expression 3 for a complete capital market in the sense that not restrictions where placed to trade with securities. Taking into account results 8a and 8b it is easy to accommodate formula 3 for the case of an incomplete market:

Lower limit:

$$E_t(R_{t+1,j}(w)) \geq R^f - \left[\frac{Cov(m_{t+1}(w), R_{t+1,j}(w))}{E_t(m_{t+1}(w))}\right] \qquad \cdots(12a)$$

Upper limit:

$$E_t(R_{t+1,j}(w)) \leq R^f - \left[\frac{Cov(m_{t+1}(w), R_{t+1,j}(w))}{E_t(m_{t+1}(w))}\right] \qquad \cdots(12b)$$

Replacing expressions 11c and 11e into expressions 12a and 12b yields the lower and the upper limit for the cost of equity capital:

$$E_t\left[R_{t+1,j}\right] \geq R_f + \frac{Cov\left(R^W_{t+1}, R_{t+1,j}\right)}{Var\left[R^W_{t+1}\right]}\left\{E_t\left[R^W_{t+1}\right] - R_f\right\} \qquad \cdots(13a)$$

[13] It is easy to obtain expression 11c, but expression 11d needs some work. In order to obtain expression 11d plug in formula 10b into formula 2b and simplify.

$$E_t\left(R_{t+1,j}\right) - R_f = \frac{Cov\left(R_{t+1}, R_{t+1,j}\right)}{Var\left(R_{t+1}\right)}\left(E_t\left(R_{t+1}\right) - R_f\right) \qquad \text{...(13b)}$$

Two observations are in order. First, note that this expression is close to the conventional CAPM in terms of the rate of return on total wealth and the risk-free rate. If the true market portfolio, which involves traded and non-traded assets, replaces the total wealth, one obtains the CAPM. Second, one may interpret this result as having two representative well-diversified investors, one for the investors not constrained and other for the constrained investors. So, taking both together there is no equilibrium in this capital market. This is the reason for a range of possible costs of equity capital.

6. Discount Rates for Non-diversified Entrepreneurs in Incomplete Markets

In the previous section it was possible to postulate the existence of two different types of well-diversified investors in the capital market. In such a context, there was not equilibrium for the market as a whole. In this section one derives a lower bound for the project discount rate assuming the existence of non-diversified entrepreneurs that actually don't trade in the capital market. In this sense, they do not have as a main concern to diversify their investment portfolio. They just seek to maximize their utility functions subject to some constraints.

Strictly speaking, one is dealing with investors' subjective beliefs and preferences that will vary from one individual to another. So, all individuals could have their own stochastic discount factor and there is nothing that leads us to think that their expectations will come ever close to each other. In this situation, it does not make sense to estimate the project added value as it were to be traded in the capital market, besides since most of these entrepreneurs are non-diversified the systematic risk is no the relevant risk anymore, but the project total risk.

One may characterize each entrepreneur preference in a different way and this is what the decision analysis technique does by estimating empirically the individual utility function. Although estimates will vary, one thinks that it is possible to use the single optimality approach with a well-known utility function to get some insights for this situation (Hoff 1997). Hence, unlike the decision analysis technique, in this section one derives an expression for the lower bound of the project's discount rate using the power utility function of the previous section, so there is no need to find out the particular utility function. However, since the coefficient of risk aversion may vary across investors, one derives two expressions corresponding to the empirical extreme values of such coefficient.

In this section, one assumes the problem faced by one entrepreneur that has the same utility function as in the previous section, but now he represents only himself. The restrictions he will face are the following:

$$C_{t+1} = W_{t+1}$$

$$W_{t+1} = R^{W}_{t+1}(W_t - C_t) \qquad \cdots(14)$$

$$R^{W}_{t+1} = R_{t+1,j}$$

In this case, the entrepreneur will put all his wealth only in one investment project, so his wealth will depend upon the return given by this single project. The problem is again to maximize his intertemporal utility function. Following the same procedure as in the previous section, the stochastic discount factor for this investor is the following:

$$m_{t+1} = a_t - b_t R_{t+1,j} \cong a_t - b_t R_{t+1,j} \qquad \cdots(15)$$

In order to obtain the values for the parameters, one needs to impose the condition that the entrepreneur could invest in the project and put some money to earn the risk-free rate; so, his SDF must fulfill both equations:

$$1 = E_t(m_{t+1}) R_f \qquad \cdots(16a)$$

$$1 = E_t(m_{t+1} R_{t+1,j}) \qquad \cdots(16b)$$

Now plugging in his stochastic discount factor (expression 15) into the above expressions yields:

$$1 = \left(a_t - b_t E_t(R_{t+1,j})\right) R_f \cong a_t = \frac{1}{R_f} + b_t E_t(R_{t+1,j}) \qquad \cdots(16c)$$

$$b_t = \frac{E_t(R_{t+1,j}) - R_f}{R_f Var(R_{t+1,j})} = \frac{1}{R_f\, \sigma(R_{t+1,j})}\left[\frac{E_t(R_{t+1,j}) - R_f}{\sigma(R_{t+1,j})}\right] \qquad \cdots(16d)$$

Looking at the second expression, one needs to decide about the value of the reward-to-variability ratio that represents the excess return that the entrepreneur requires in order to take one more unit of risk.

As it is shown below, this ratio depends upon the entrepreneur's optimal choice among risk-free deposits and investing in the project, his coefficient of risk aversion and the project total risk. If one assumes that the optimal choice for the investor is to put all his capital into the project, one is dealing with the least risk-averse investor. Further, if one assumes a coefficient of risk-aversion of 2 for the entrepreneur, and a project's total risk equal or higher than 50%, then the reward-to-variability ratio must be equal or higher than one[14].

Taking this value for the reward-to-variability ratio (1), expressions 16c and 16d can simplify to:

$$a_t = \frac{1}{R_f} + \left(\frac{1}{R_f}\right)\frac{E_t(R_{t+1,j})}{\sigma R_{t+1,j}} \qquad \cdots(16e)$$

[14] Several studies have reported coefficients of risk aversion for the representative investor in the range between 2 and 4 (Bodie *et al.* 2002). One may use this range as a first approximation.

$$\lambda_t = \frac{1}{R_f \sqrt{V(R_{t+1,j})}} \quad \cdots(16f)$$

Finally, if one replaces expressions 16e and 16f into formula 12a one gets the lower limit for the project hurdle rate:

$$E_t[R_{t+1,j}] \geq R_f + \sqrt{V(R_{t+1,j})} \quad \cdots(17a)$$

A natural question is the following: What would happen if the reward-to-variability ratio were two instead of one? In this case, the lower limit will increase:

$$E_t[R_{t+1,j}] \geq R_f + 2\sqrt{V(R_{t+1,j})} \quad \cdots(17b)$$

As one may see, the expression for the project hurdle rate is sensitive to the reward-to-variability ratio—therefore it is important to explain what determines this ratio. In order to gain some idea, let's assume that the entrepreneur has the following utility function in terms of expected returns and return variability (Bodie et al., 2002):

$$U = E(r) - 0.005A\sigma^2 \quad \cdots(18)$$

In this expression, 0.005 is a scale factor and "A" represents the entrepreneur's coefficient of risk aversion. The expected return and variance of the portfolio that comprises the risk-free rate and investment project is as follows:

$$E(R_p) = R_f + y[E(R_{t+1,j}) - R_f]$$

$$\sigma_p^2 = y^2\sigma^2(R_{t+1,j})$$

Hence, the entrepreneur wishes to maximize the following utility function:

$$\max U = R_f + y[E_t(R_{t+1,j}) - R_f] - 0.005A\, y^2\sigma^2(R_{t+1,j})$$

The optimal position for this entrepreneur is the following[15]:

$$y^* = \left[\frac{1}{(0.01)(A)\sigma(R_{t+1,j})}\right]\left[\frac{E(R_{t+1,j}) - R_f}{\sigma(R_{t+1,j})}\right] \quad \cdots(19)$$

The optimal position for this entrepreneur depends on the reward-to-variability ratio and his coefficient of risk aversion. Solving for the reward-to-variability ratio, one concludes that it depends on the optimal position for the entrepreneur, his coefficient of risk aversion and the project's total risk. For instance, for the least risk-averse entrepreneur (A=2) that invests only in the investment project subject to total risk of 25%, the reward-to-variability ratio must be 0.50; while for the most risk-averse entrepreneur (A=4) the reward-to-variability ratio would be 1. Whenever the project's total risk increases, the reward-to-variability ratio will also

[15] The number 0.01 is a scale factor because one uses percentages instead of decimals.

increase. For instance, if the project total risk is 50% and the optimal choice for the investor is to invest only in the project, the reward-to variability ratio will range between 1 (for A=2) and 2 (for A=4).

In general, the reward-to-variability ratio depends positively on the project's total risk for a given coefficient of risk aversion. In this sense, if the project total risk changes through time, the reward-to-variability ratio must also change proportionally. Therefore, a crucial issue to estimate the project's hurdle rate is the estimation of the project's total risk[16].

7. Conclusion

The previous results can be applied in the context of an emerging market. Three types of investors operate in these markets: Globally diversified investors, well-diversified local investors, and non-diversified entrepreneurs. However, the first two groups are really scarce in emerging markets. Furthermore, local investors are not usually diversified, because local institutional investors usually dominate local investments and they are restricted in their investments. For instance, among the biggest institutional investors in Peru are Pension Funds, but they are restricted in their investment abroad by law.

Nevertheless, for the few local well-diversified investors, the result is that there is no such thing as a unique cost of equity capital for an investment project even assuming a completely segmented market. In fact, there could be different types of cost of equity capital. A crucial issue is how to estimate the model parameters in the context of an emerging market, where a hybrid model of partial integration is intuitively more suitable. This is an unsolved issue, subject to future research.

The result concerning non-diversified entrepreneurs is applicable in emerging markets. The result does not rely upon stock return distributions, also not upon the assumption of a complete capital market. Furthermore, it is designed to non-diversified entrepreneurs who are the vast majority of investors in these markets. Non-diversified entrepreneurs usually put all their savings in one enterprise without caring about diversification. Advocates of using a country risk premia may find that expressions 17a and 17b can allow for country risk in the estimation of the project's total risk. using scenario analysis.

However, one must be prepared to accept the subjective nature of the analysis. In fact, to get some estimate of the project total risk, one must perform a subjective valuation of the investment project. The process of risk analysis must go through sensitive analysis to detect critical variables; eliciting expert opinions to characterize the critical variables; and simulate the variables within each scenario to obtain the project's total risk per period. This is a process where some bias may arise, so one must be careful with the model inputs. Despite these cautions, one believes that this approach is a useful way to estimate project's hurdle rates for non-diversified entrepreneurs in emerging markets and also in countries where there is no capital market.

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[16] Mongrut (2004b) proposes a methodology in order to estimate the project's total risk for non-diversified entrepreneurs.

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