A Sufficient Condition for Synchronization Risk and Delayed Arbitrage

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This paper examines the sufficient condition for the existence of synchronization risk as defined in Abreu and Brunnermeier (2003). Using a numerical example, it shows that there is an upper bound to the selling threshold for bubble bursting. This implies that the selling threshold stipulated as an exogenous variable in Abreu and Brunnermeier (2003) should instead be treated endogenously.

Introduction

This paper attempts to analyze the sufficient condition of the model presented in Abreu and Brunnermeier (2003). Abreu and Brunnermeier's (2003) model assumes that arbitrageurs face synchronization risk as defined in Abreu and Brunnermeier (2002), and delay the use of arbitrage opportunities. Rational arbitrageurs, sequentially informed about the bubble bursting are unable to synchronize their activities and sell at the same time. Abreu and Brunnermeier (2003)[1] concluded that the bubble persists because of the delay in arbitrage.[2]

A number of recent empirical and theoretical studies have examined the existence of synchronization risk. Brunnermeier and Nagel (2004) empirically showed the existence of synchronization risk during the US technology bubble. Sakawa and Watanabel (2006) showed the sufficient condition for the existence of synchronization risk in a discrete time setting of Abreu and Brunnermeier's (2003) model.

This paper re-examines the sufficient condition for the existence of synchronization risk in a discrete time setting. It focuses on two points. First, it finds the sufficient condition of synchronization risk that depends on the exogenous threshold value of bubble bursting. Second, it shows that the smaller the trading interval and number of arbitrageurs in the market, the larger the exogenous threshold value under the sufficient condition.

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1 Chamley (2004), surveys their paper.

2 Blanchard (1979) and Blanchard and Watson (1982) analyze the mechanisms of bubble persistence and crashing. Their models conclude that the crash occurs stochastically under the no-arbitrage condition.

The Basic Setting

We introduce the discrete time approximation of Abreu and Brunnermeier's (2003) model as in Sakawa and Watanabel (2006). We represent discrete time as $0 = t_0 < t_1 < ... < t_n$, and the length of one trading period as $\Delta\ (\Delta = t_{i+1} - t_i,\ i \in [1, n])$. There are m risk-neutral arbitrageurs and only one risky stock in the market (all arbitrageurs are assumed rational and m is a natural number). Each arbitrageur has $(1/m)$ units of stock.

At time t_b, a positive shock occurs and a bubble emerges. Before time t_b, the price of the stock (ψ_i) was equal to its fundamental price (υ_i). At time t_b, the shock occurs and the growth rate of fundamental price is adjusted to the safe interest rate (r). Therefore, the bubble emerges after time $t_b, \psi_i = (1+g)^{\{(i-b)\Delta\}} > \upsilon_i = (1+r)^{\{(i-b)\Delta\}}$. This model treats time t_b as a random variable.

Between t_{b+1} and $t_{b+m}\ (= t_b + \eta)$, the rational arbitrageurs become sequentially aware of the new fundamental value at a uniform rate. In other words, it takes Δ to be informed of one arbitrageur, and $\eta = m\Delta$ exists. An individual arbitrageur who becomes aware of the change in fundamentals at time t_j, believes that t_b is distributed between t_{j-m} and t_{j-1}. We denote this type of arbitrageur by j. If $b = j - 1$, this arbitrageur is the first to realize that the fundamental value has deviated from the stock price. If $b = j - m$, all other traders have already received this information. Arbitrageur j does not know the time when the shock has occurred. Arbitrageur j sells his stock for an arbitrage profit, after he receives the information regarding shock. As the information regarding shock arrives at a uniform rate, the number of arbitrageurs selling orders gradually increases, but the selling is not synchronized.

The Analysis of Abreu and Brunnermeier's (2003) Discrete Time Model

We define the distribution function of the crash probability as $\phi(t_{j+\tau} \mid j)$,[3] and the transaction cost in this market as $C(<1)$. Arbitrageur j is faced with the problem of maximizing his expected profit as follows:

$$\text{Max} \sum_{s=0}^{\tau-1} \left\{ \left(\frac{1}{1+\mathrm{r}}\right)^{(j+r-b)\Delta} \big(1-\beta(j+s-b)\big)(1+g)^{(j+r-b)\Delta} \big(1-\phi(t_{j+\tau} \mid j)\big) \right\}$$

$$+ \left(\frac{1}{1+r}\right)^{(j+r-b)\Delta} (1+g)^{(j+r-b)\Delta} \big(1-\phi(t_{j+\tau} \mid j)\big) - C$$

3 Arbitrageur j's belief is represented by the conditional probability density function: $\phi(t_{j+\tau} \mid j)$. $\phi(t_{j+\tau} \mid j)$ is a monotonically increasing function of τ. All arbitrageurs believe that the bubble persists when $\phi(t_{j+\tau} \mid j)$ becomes zero.

Arbitrageur *j*'s expected profit is maximized when the increase in the first term is equal to the decrease in the second term.[4] The arbitrage profit is zero when the optimal strategy[5] ($\tau = \tau^1$) is undertaken. As a result, the no-arbitrage condition is defined using the hazard rate: $h\left(t_{j+\tau} \mid j\right)$.

$$h\left(t_{j+\tau} \mid j\right) = \frac{\phi\left(t_{j+r} \mid j\right)}{1+\Phi\left(t_{j+r} \mid j\right)} = \frac{(1+g)^{\Delta} - (1+r)^{\Delta}}{\beta\left(j+\tau^1-b\right)(1+g)^{\Delta}} \qquad ...(1)$$

When the arbitrageurs undertake the trigger strategy, they begin to sell out the stock after time $t_{b+1+\tau^1}$.[6] We define time, $t_{b+\tau}$ as the time when the selling pressure crosses $\left(\kappa : \frac{\left(\tau^* - \tau^1\right)\Delta}{\eta} = \kappa\right)$. Arbitrageur *j* continues to hold the stock until he believes that the capital gain from holding the stock exceeds the expected loss from the crash.

In a symmetric equilibrium, the bubble bursts when arbitrageur $b + m\kappa$ learns about the mispricing. $j = b + m\kappa$ is uniquely determined under the perfect Bayesian Nash equilibrium, and we consider the prior distribution as a geometric distribution function.[7] At time t_b, arbitrageur *j*'s prior distribution function ($\Phi(t_b)$) and density function ($\phi(t_b)$) are given as follows:[8]

$$\Phi(t_b) = 1 - p^b, \phi(t_b) = (1-p)p^b, \; 0 < p < 1 \qquad ...(2)$$

We assume that the arbitrageur *j* estimates that the bursting time is $t_{b+\zeta}$. As the bubble bursts at time, $t_{j+\tau}$ when he sells out the stock, the following equation is established:

$$(b+\zeta)\,\Delta = (j+\tau)\,\Delta \qquad (b = j+\tau-\zeta) \qquad ...(3)$$

Because the bubble bursts when arbitrageur $b + m\kappa$ learns about the mispricing in equilibrium, the variable ζ that represents the bursting time becomes: $\zeta = m\kappa + \tau$. The optimal trading time, $t_{j+\tau^1}$ is the solution to Equation (1). The bubble bursts when arbitrageur $\tau^1 + m\kappa$ sells out the stock at time, $b+\tau^1+m\kappa$ in equilibrium. Arbitrageur τ^1's optimal strategy then becomes:[9]

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4 The first term is an increasing function of τ while the second term is a decreasing function because the probability of a crash is an increasing function.

5 The induction is referred to in Sakawa and Watanabel (2006).

6 Sakawa and Watanabel (2006) prove the optimality of this strategy.

7 Abreu and Brunnermeier (2003) define the prior distribution as an exponential distribution function. Sakawa and Watanabel (2006) use the geometric distribution function with $p = 1/2$.

8 The posterior distribution function: $\Phi\left(t_b \mid j\right)$ is calculated as, $\frac{\sum_{j=i-m}^{b-1} \phi(t_j)}{\Phi\left(t_{j-1}\right) - \Phi\left(t_{j-m}\right)}$. Arbitrageur *j* estimates that the bubble bursts during the interval, $t_{j-m} \le t_b \le t_{j-1}$ after receiving information about the burst at t_j. Therefore, $\Phi\left(t_b \mid j\right)$ follows the truncated distribution function with the interval, $t_{j-m} \le t_b \le t_{j-1}$.

9 The Appendix discusses how to induce this equation.

$$\tau^1 = \frac{\ln(1+g) - \ln\left[(1+g)^{\Delta} - \left(p^{-1} - p^{m\kappa-2}\right)\left\{(1+g)^{\Delta} - (1+r)^{\Delta}\right\}\right]}{\Delta\left\{\ln(1+g) - \ln(1+r)\right\}} - m\kappa \qquad \text{...(4)}$$

Therefore, the sufficient condition becomes a positive value of τ^1 in Equation (4).

An Examination of the Sufficient Condition

In the preceding section, we specify the arbitrageurs' optimal strategy τ^1 by Equation (4), using a geometric function. Sakawa and Watanabel (2006) showed using a numerical example, that synchronization risk does not exist when $\eta(=m\Delta)$ exceeds an upper bound. Their paper does not analyze whether the existence of synchronization risk depends on the value of the probability of burst (p) and the threshold (κ). We check the sufficient condition of the threshold (κ) to satisfy the existence of synchronization risk for any value of the probability of burst (p).

The sufficient condition is satisfied when τ^1 is positive for any/all p ($0 < p < 1$). Considering the sufficient condition, the following proposition is established:

Proposition 1*: The optimal τ^1 is positive for any/all p ($0 < p < 1$), when the following condition is satisfied:*

$$\frac{1}{m} < \kappa < \frac{(1-\Delta)\ln(1+g)}{m\Delta\left\{\ln(1+g) - \ln(1+r)\right\}}$$

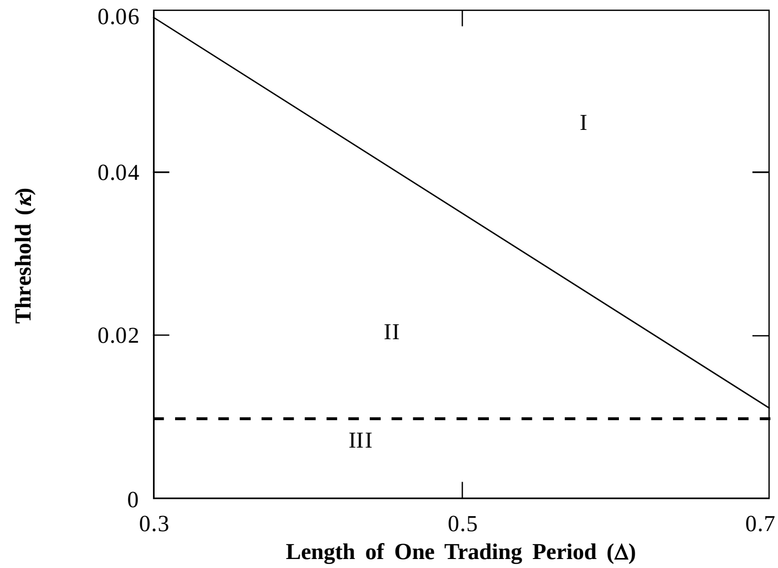
Proposition 1 shows the sufficient condition for threshold (κ) for any probability of burst (p). In Proposition 1, the left hand side of the inequality implies that exogenous selling threshold (κ) goes beyond the selling units of stock of one arbitrageur ($1/m$) in equilibrium. In other words, it implies that bubble bursts after arbitrageur, $b+m\kappa\,(> b+1)$ learns about the mispricing in equilibrium. The right hand side of the inequality is equivalent to $m\kappa < \dfrac{(1-\Delta)\ln(1+g)}{\Delta\left\{\ln(1+g)-\ln(1+r)\right\}}$ and shows the upper bound of the threshold (κ). This condition is expressed as a numerical example in Figures 1 and 2.

Solving the relation between threshold (κ) and time interval (Δ), the numerical values obtained are: $g = 0.05$, $r = 0.03$, and $m = 100$. The horizontal axis in Figure 1 represents the time interval (Δ) and the vertical axis represents the threshold (κ). The solid line shows the upper bound of the threshold (κ) in Area I and dotted line shows the lower bound in Area III. So, Area II represents the range of threshold (κ) that satisfies the sufficient condition. Figure 1 shows that there is an upper bound of the threshold (κ) satisfying the existence of synchronization risk and the sufficient condition is likely to be satisfied in smaller trading time interval.

On the other hand, Figure 2 shows the relation between threshold (κ) and the number of arbitrageurs (m). The numerical values obtained are: $g = 0.05$, $r = 0.03$, and $\Delta = 0.3$.

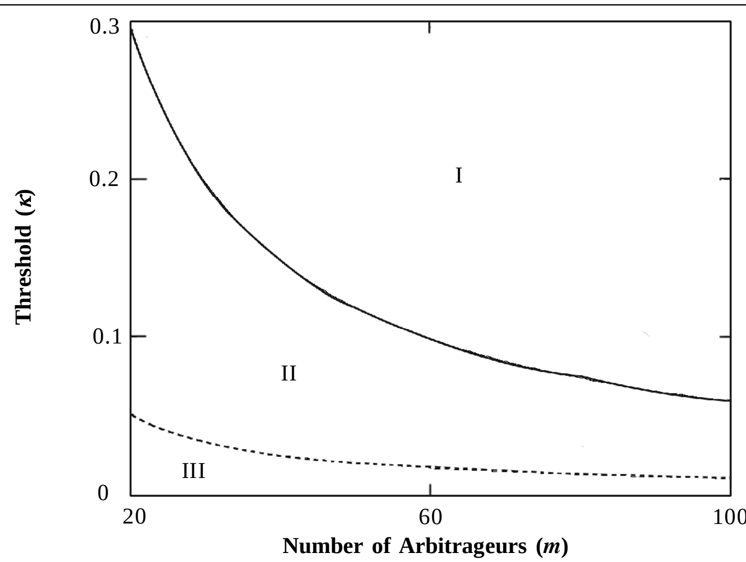
In Area II of Figure 2, we can see that the larger the number of arbitrageurs (m) in the market, the smaller the threshold (κ).

Figure 1: A Numerical Example Indicating the Relationship Between the Length of One Trading Period (Δ) and the Threshold (κ)



Note: g = 0.05; r = 0.03; m = 100.

Figure 2: A Numerical Example Indicating the Relationship Between Number of Arbitrageurs (m) and the Threshold (κ)



Note: g = 0.05; r = 0.03; Δ = 0.3.

Conclusion

In this paper, two main inferences have been drawn about the sufficient condition found in Abreu and Brunnermeier (2003). First, the sufficient condition for the existence of synchronization risk was re-examined using numerical examples. The existence of synchronization risk is found to depend on the value of the exogenous threshold of bubble bursting (κ). Second, the smaller the trading interval (Δ) and number of arbitrageurs (m) in the market, the larger the exogenous threshold value (κ) under the sufficient condition.❖

Acknowledgment: *The authors are grateful to Yoshiro Tsutsui, Akira Kohsaka, Shinsuke Ikeda, and Masamitsu Onishi for their valuable comments on an earlier draft.*

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Appendix

Induction of Equation (4)

By substituting Equation (3) into Equation (1), the necessary conditions of the symmetric Bayesian Nash equilibrium become:

$$h\left(t_{j+\tau} \mid j\right)=\frac{\phi\left(t_{j+r} \mid j\right)}{1-\Phi\left(t_{j+r} \mid j\right)}=\frac{(1+g)^{\Delta}-(1+r)^{\Delta}}{\beta\left(\tau^{1}+m\kappa\right)(1+g)^{\Delta}}$$

(Cont.)

Appendix

By solving the above equation, arbitrageur $\tau^1 + m\kappa$'s optimal strategy is obtained as follows:

$$\left(\tau^1 + m\kappa\right)\Delta = \frac{\ln(1+g) - \ln\left[(1+g)^\Delta - \left(p^{-1} - p^{m\kappa-2}\right)\left\{(1+g)^\Delta - (1+r)^\Delta\right\}\right]}{\left\{\ln(1+g) - \ln(1+r)\right\}}$$

In an endogenous crash, selling pressure crosses $\kappa(<1)$ at $t_{b+\tau}$, and $\tau^*\Delta = \left(\tau^1 + m\kappa\right)\Delta$ exists. So, we solve $t_{b+\tau}$ and $t_{b+\tau^1}$ as follows:

$$\tau^* = \frac{\ln(1+g) - \ln\left[(1+g)^\Delta - \left(p^{-1} - p^{m\kappa-2}\right)\left\{(1+g)^\Delta - (1+r)^\Delta\right\}\right]}{\Delta\left\{\ln(1+g) - \ln(1+r)\right\}}$$

Proof of Proposition 1: By differentiating Equation (4), the following relation between τ^1 and p is established:

$$\frac{\partial \tau^1}{\partial p} = -\frac{1 + (m\kappa - 2)\,p^{m\kappa-1}}{p^2} < 0$$

The optimal τ^1 becomes a decreasing function for any $p \in (0,1)$ under the condition, $m\kappa > 1$ in the above equation.

Therefore, the optimal τ^1 is positive for any $p \in (0,1)$, when $\tau^1(p=1) > 0$ exists.

$$\left\{\tau^1(p=1)\right\} > 0 \Leftrightarrow \kappa < \frac{(1-\Delta)\ln(1+g)}{m\Delta\left\{\ln(1+g) - \ln(1+r)\right\}}$$

So, the positive τ^1 for any $p \in (0,1)$ is satisfied under the following condition:

$$\frac{1}{m} < \kappa < \frac{(1-\Delta)\ln(1+g)}{m\Delta\left\{\ln(1+g) - \ln(1+r)\right\}}$$

Reference # 42J-2009-06-02-01