CAPM and Capital Budgeting: Present/Future, Equilibrium/ Disequilibrium, Decision/Valuation

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This paper expands on the results obtained in Magni (2009) regarding investment decisions with the Capital Asset Pricing Model (CAPM). It is shown that four different decision criteria are deductively drawn from this model: the disequilibrium Net Present Value (NPV), the equilibrium NPV, the disequilibrium Net Future Value (NFV), and the equilibrium NFV. It is shown that all of them may be used for accept-reject decisions, but only the equilibrium NPV and the disequilibrium NFV may be used for valuation, given that they have the additivity property. However, it is possible to deductively dismiss the two nonadditive indexes if the 'accept/reject' problem is reframed as a choice among mutually exclusive alternatives. As for the remaining (additive) measures, the equilibrium NPV and the disequilibrium NFV are unreliable for both valuation and decision, because despite their additivity, they do not signal arbitrage opportunities whenever there is some state of nature for which they are decreasing functions with respect to the end-of-period cash flow. In this case, the equilibrium value of a project is not the price it would have if it was traded in the security market. This result is the capital-budgeting counterpart of Dybvig and Ingersoll's (1982) result.

Introduction

The use of the Net Present Value (NPV) as a tool for investment decisions and valuation is widespread in finance and management science. In construction industry, project scheduling problems, production inventory problems, advance manufacturing technology, and logistics, scholars usually consider, "the maximization of the NPV of the project, as the more appropriate objective" (Herroelen *et al.*, 1997, p. 97). The NPV as a tool for decision making has been employed in these fields for several decades (Doersch and Patterson, 1977; Smith-Daniels and Smith-Daniels, 1987; Elmaghraby and Herroelen, 1990; Yang *et al.*, 1993; Pinder and Maruchek, 1996; Etgar *et al.*, 1997; Kimms, 2001; Najafi and Niaki, 2005; De Reyck *et al.*, 2008; Hsieh *et al.*, 2008; and Sobel *et al.*, 2009). Very scarce is the literature that directly deals with the way a discount rate should be computed in case of uncertainty (Magni, 2002 and 2009; De Reyck, 2005; and De Reyck *et al.*, 2008).

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This paper aims at investigating that particular version of the NPV which makes use of the Capital Asset Pricing Model (CAPM) for computing the discount rate. The thesis of this work is not that the NPV is fallacious in itself, but that the use of the CAPM-based NPV brings about some nontrivial problems. The use of the CAPM for capital budgeting purposes actually goes back to the 1960s and 1970s, when various authors developed a theoretical link between this asset pricing model and corporate capital budgeting decisions. Among the several contributions, we find classical papers of foremost authorities, such as Tuttle and Litzenberger (1968), Hamada (1969), Mossin (1969), Litzenberger and Budd (1970), Stapleton (1971 and 1974), Rubinstein (1973), Bierman and Hass (1973 and 1974) and Bogue and Roll (1974). The decision criteria presented by these authors are seemingly different, but logically, they are equivalent (Senbet and Thompson, 1978) and may be framed in terms of risk-adjusted cost of capital (Magni, 2007a and 2009). The resulting capital budgeting criterion suggests that, as long as the CAPM assumptions are met, a firm aiming at maximizing share price should undertake a project if and only if the project's expected internal rate of return exceeds the project's risk-adjusted cost of capital. These classical papers are aimed at formally deducting a decision rule from the CAPM, but do not particularly focus on project valuation. Although the NPV rule is often reminded, no explicit claim appears on whether the risk-adjusted cost of capital may or may not be used for valuing projects. As a result, ambiguities arise on the use of the project NPV as a decision rule or as a valuation tool. The risk-adjusted cost of capital is presented as depending on a disequilibrium (cost-based) systematic risk (Rubinstein, 1973), but project value is often framed in a certainty-equivalent form (Bogue and Roll, 1974), which implies that an equilibrium systematic risk is used. Therefore, other uncertainties are added, regarding the use of either the equilibrium systematic risk (equilibrium NPV) or the disequilibrium systematic risk (disequilibrium NPV).

A few contributions have drawn attention on these topics. Among these, we find Rendleman's (1978) paper, which deals with the use of cost-based (disequilibrium) covariance terms as opposed to market-determined (equilibrium) covariance terms. The author suggests that if a firm were to rank projects on the basis of excess of internal return over equilibrium (market-determined) return, an incorrect decision would be reached. Haley and Schall (1979, pp. 182-183) show that the disequilibrium NPV is unreliable in ranking projects. Weston and Chen (1980) state that either the disequilibrium or equilibrium return may be used for ranking projects, if appropriate use is made of both. And while the equilibrium form of NPV is widespread for valuation purposes (in the classical certainty-equivalent form), the disequilibrium form of NPV has its own upholders as well among scholars. For example, Lewellen (1977) uses the disequilibrium NPV to value projects; Copeland and Weston use cost-based betas, and therefore disequilibrium NPVs, for valuing projects in various occasions (Copeland and Weston, 1983 and 1988; and Weston and Copeland, 1988); Bossaerts and Ødegaard (2001) endorse the use of the disequilibrium NPV for valuing projects, but in their subsequent edition they maintain that the equilibrium NPV is the correct NPV (Bossaerts and Ødegaard, 2006). Some other authors are aware that the disequilibrium NPV is often used in finance, and warn against it, claiming that this kind of NPV is a common misuse of the NPV rule. Ang and Lewellen (1982, p. 9) explicitly claim that the disequilibrium NPV is the 'standard discounting approach' in finance for valuing projects, and show that such a method is incorrect for it leads to nonadditive valuations. Grinblatt and Titman (1998), being aware that the use of disequilibrium NPVs is extensive, present an example where cost-based betas are used (see their Example 10.5) and claim that their example deliberately shows an incorrect procedure. Ekern (2006) properly distinguishes between NPV as a decision rule and NPV as a valuation tool. He states that the disequilibrium NPV is correct for decision but not for valuation, and suggests the use of the equilibrium NPV as well as several other equivalent methods. Magni (2007b) focuses on the relation between disequilibrium NPV and absence of arbitrage, showing that while deductively valid as a decision tool, the former is incompatible with the latter. The issue is so subtle that some debates arise from misunderstanding. For example, Magni (2002) attacks the use of the CAPM in its disequilibrium form and De Reyck (2005) opines that the use of CAPM is correct for investment decisions, but takes for granted that the equilibrium version of the CAPM is the correct way of using CAPM (the misunderstanding conceals the fact that both the authors agree that the disequilibrium NPV is incorrect for valuation).

In addition to the dichotomies decision/valuation, equilibrium/disequilibrium, a third dichotomy is completely neglected. No thorough study exists on relations between present values and future values under uncertainty. The notion of future value is important, though overwhelmed by the notion of present value. Under uncertainty, future values and present values are equivalent concepts, referred to different times—"Most frequently it is implicitly assumed that the objective is to maximize the present value of the firm or to maximize the future value of the firm at some particular point in time" (Teichroew *et al.*, 1965b, p. 152). The future value is useful because, "it leads naturally to the concept of the project balance (Teichroew *et al.*, 1965b, p. 155), and the firm's excess return (Young and O'Byrne, 2001) is just the firm's Net Future Value (NFV). The notion of residual income bears strict relation to the notion of NFV (Magni, 2009), and the classical investment criterion proposed by Teichroew *et al.* (1965a and 1965b) is developed starting from future value functions. A brief paper by Weston and Chen (1980) is the only one, to the best of our knowledge, that jointly tackles the problem of present/future value along with the problem of equilibrium/ disequilibrium systematic risk.

This paper, limiting its scope to one-period projects and accept-reject situations, aims at giving some clarification on these topics. In particular, this work is a natural development of Magni's (2009) paper. In the latter, it is shown that the use of a cost-based (i.e., disequilibrium) required rate of return is widespread in corporate finance, and leads to a disequilibrium NPV that, while legitimate for decision purposes, may not be used for valuation purposes. While the paper deductively draws the disequilibrium NPV, it does not investigate the derivation of NFVs, either in equilibrium or disequilibrium form. Also, it dismisses the disequilibrium NPV by invoking additivity as a rational tenet. In this work, it is shown that four decision criteria may be derived from the CAPM and that the nonadditive measures may be dismissed by properly framing the decision problem. In particular, it is shown that:

- If the CAPM assumptions are met in the security market and a firm's objective is to maximize share price, the investor may reliably employ either present or future values, either in equilibrium or disequilibrium format, as long as the resulting values are used for decision-making purposes. If, instead, the purpose is valuation, only the disequilibrium NFV and the equilibrium NPV may be used, because the disequilibrium NPV and the equilibrium NFV are not additive.

- While legitimate as decision rules, nonadditivity makes the latter unsafe: whenever decision makers face a portfolio of projects (or a project composed of several subprojects) they may separately compute each project's NPV (NFV) and then sum the values obtained, or sum the cash flows and then compute the portfolio NPV. By changing the order in which summation and discounting are made, different results are obtained. This result is a conundrum, because the two nonadditive indexes are validly deducted from the CAPM assumptions.

- The two nonadditive indexes cannot be deducted from the CAPM assumption if the decision problem is reframed: instead of coping with the problem 'accept *Z*/reject *Z*' one may consider the problem 'invest in *Z*/invest in *Y*', where *Y* is an alternative course of action. The latter case is more general and reduces to the former case whenever *Y* is the null alternative, that is, the project with zero cash flows. As a result, the equilibrium NPV and the disequilibrium NFV are the only capital budgeting criteria that are validly deducted from the CAPM.

- Yet, the latter have serious pitfalls as well: if there is a state of nature for which they are decreasing functions with respect to the end-of-period cash flow, then valuation (and decision) is unreliable. This result is just the capital-budgeting version of a result found in Dybvig and Ingersoll (1982) concerning asset pricing in complete markets, and explains why the equilibrium value of a project is not always the price it would have if it was traded in the security market.

The paper is structured as follows: it presents the definitions of NPVs and NFVs, in both equilibrium and disequilibrium format, followed by formal deduction of four decision criteria assuming that the CAPM assumptions are met. Subsequently, the equilibrium NPV and the disequilibrium NFV are shown to be additive, whereas the disequilibrium NPV and the equilibrium NFV are shown to be nonadditive, and it is also shown that by reframing the decision problem, the nonadditive measures can be dismissed. It also shows that additivity does not guarantee absence of arbitrage and thus the two additive measures previously obtained may be in some cases misleading. Finally, it is shown that the equilibrium value of a project is not necessarily the value that the project would have if it was traded in the security market, followed by the conclusion.

Equilibrium in the security market is assumed throughout the paper, unless otherwise specified. To avoid pedantry, main notational conventions are presented in Table 1.

Table 1: Main Notational Conventions	
$F_j\left(\bar{F}_j\right)$	Asset j's end-of-period random (expected) cash flow
I_j	Cost of project j
$r_j\left(\bar{r}_j\right)$	Asset j's random (expected) rate of return
$V_j^e\left(V_j^d\right)$	Equilibrium (Disequilibrium) value of asset j
r_Z^d	Disequilibrium (cost-based) rate of return of project Z (a.k.a. risk-adjusted cost of capital)
$r_Z^e\left(r_Y^e\right)$	Equilibrium rate of return of project $Z(Y)$
$r_f(R_f)$	Risk-free rate (1 + risk-free rate)
σ_m^2	Variance of the market rate of return
cov	Covariance
$\lambda := \frac{\bar{r}_m - r_f}{\sigma_m^2}$	Market price of risk
P_l	Price of firm l's shares before acceptance of the project
$P_l^{\circ}\left(P_l^{\bullet}\right)$	Price of firm l's shares before acceptance of project $Z(Y)$
N_l	Number of firm l's outstanding shares
$N_l^{\circ}\left(N_l^{\bullet}\right)$	Additional shares issued at price $P_l^{\circ}\left(P_l^{\bullet}\right)$ to finance project $Z(Y)$
V_l	Firm value before acceptance of the project
V_l°	Firm value after acceptance of project Z
$dNPV_j$, $eNPV_j$	Disequilibrium (Equilibrium) net present value of project j
$dNFV_j$, $eNFV_j$	Disequilibrium (Equilibrium) net future value of project j
	$j = Z, Y, Z_1, Z_2, l, m$

Equilibrium and Disequilibrium, Present and Future

This section introduces the notions of NPV and NFV and shows that, under uncertainty, they are not univocal.

Under certainty, NPV and NFV are equivalent notions. In particular, let $V_Z = F_Z/(1 + i)$ be the project's value, where i is the (opportunity) cost of capital. The NPV of a project Z with cost I_Z and end-of-period cash flow F_Z is given by:

$$NPV_Z = -I_Z + V_Z = -I_Z + \frac{F_Z}{(1+i)} \quad ...(1)$$

The NFV of project Z is just the NPV compounded at the cost of capital:

$$NFV_Z = NPV_Z\left(1+i\right) = -I_Z\left(1+i\right) + F_Z \quad ...(2)$$

As $r_Z = F_Z/I_Z - 1$ is the project rate of return, the NFV may be rewritten in excess-return form as:

$$\text{Excess Return} = I_Z(r_Z - i) = NFV_Z \quad ...(3)$$

Therefore, the NPV is just the present value of the project's excess return, calculated at the cost of capital:

$$NPV_Z = \frac{I_Z\left(r_Z - i\right)}{(1+i)} \quad ...(4)$$

Under certainty, the NPV is the current project (net) value, the NFV (excess return) is the end-of-period project (net) value. In terms of decisions, the NPV and the NFV have the same sign (as long as $(1 + i) > 0$), so that a project is worth undertaking if and only if the NPV and the NFV are positive. The NPV and NFV are twin notions—both may interchangeably be used as decision rules and valuation tools.

Under uncertainty, if the CAPM is used for measuring risk, the notions of NPV and NFV (and the very notion of value) are not univocal. Depending on whether disequilibrium covariance terms or equilibrium covariance terms are used, we find disequilibrium or equilibrium NPVs and NFVs, and then give the following definitions:

Definition 1: The disequilibrium NPV (dNPV) is the net discounted expected cash flow, where the discount rate is the disequilibrium (cost-based) rate of return of the project, $r_Z^d = r_f + \lambda \operatorname{cov}\left(F_Z, r_m\right)/I_Z$:

$$dNPV_Z := \frac{\overline{F}_Z}{R_f + \dfrac{\lambda}{I_Z}\operatorname{cov}\left(F_Z, r_m\right)} - I_Z \quad ...(5)$$

The first addend is the disequilibrium value of the project, so that $dNPV_Z := V_Z^d - I_Z$.

Definition 2: The equilibrium NPV (eNPV) is the net discounted expected cash flow, where the discount rate is the equilibrium rate of return, $r_Z^e = r_f + \lambda \operatorname{cov}\left(F_Z, r_m\right)/V_Z^e$ (with V_Z^e being the equilibrium value of the project),

$$eNPV_Z := \frac{\overline{F}_Z}{R_f + \frac{\lambda}{V_Z^e}\operatorname{cov}(F_Z, r_m)} - I_Z \qquad ...(6)$$

As widely known, we have $V_Z^e := \left(\overline{F}_Z - \lambda \operatorname{cov}(F_Z, r_m)\right)/R_f$ so that we may alternatively reframe the eNPV in a certainty-equivalent form as:

$$eNPV_Z := \frac{\overline{F}_Z - \lambda \operatorname{cov}(F_Z, r_m)}{R_f} - I_Z \qquad ...(7)$$

Definition 3: The disequilibrium NFV (dNFV) is given by the compounded disequilibrium NPV: $dNFV_Z = dNPV_Z\left(1 + r_Z^d\right) = dNPV_Z\left(R_f + \lambda \operatorname{cov}(F_Z, r_m)/I_Z\right)$. Therefore, we may write, in an excess-return format as,

$$dNFV_Z := I_Z\left(\overline{r}_Z - r_Z^d\right) = I_Z\left(\overline{r}_Z - r_f - \frac{\lambda \operatorname{cov}(F_Z, r_m)}{I_Z}\right) \qquad ...(8)$$

Definition 4: The equilibrium NFV (eNFV) is given by the compounded $eNPV_Z$, $eNFV_Z = eNPV_Z\left(1 + r_Z^e\right)$. Therefore, we may write, in an excess-return format as,

$$eNFV_Z = I_Z\left(\overline{r}_Z - r_Z^e\right) = I_Z\left(\overline{r}_Z - r_f - \frac{\lambda \operatorname{cov}(F_Z, r_m)}{V_Z^e}\right) \qquad ...(9)$$

or, using the relation $\overline{F}_Z - I_Z = \overline{r}_Z\, I_Z$,

$$eNFV_Z = \left(\overline{F}_Z - I_Z\right) - \left(r_f + \frac{\lambda \operatorname{cov}(F_Z, r_m)}{V_Z^e}\right) I_Z \qquad ...(10)$$

Remark 1: It is worth reminding that the project's expected rate of return differs from both the disequilibrium rate of return and the equilibrium rate of return. For the sake of clarity, the three rates of return may be written as (see also Weston and Chen, 1980, p. 12):

$$\overline{r}_Z = \frac{\overline{F}_Z}{I_Z} - 1 \qquad \text{Expected rate of return} \qquad ...(11)$$

$$r_Z^d = \frac{\overline{F}_Z}{V_Z^d} - 1 = r_f + \frac{\lambda \operatorname{cov}(F_Z, r_m)}{I_Z} \qquad \text{Disequilibrium rate of return} \qquad ...(12)$$

$$r_Z^e = \frac{\overline{F}_Z}{V_Z^e} - 1 = r_f + \frac{\lambda \operatorname{cov}(F_Z, r_m)}{V_Z^e} \qquad \text{Equilibrium rate of return} \qquad ...(13)$$

The disequilibrium rate of return in Equation (12) is the risk-adjusted cost of capital introduced in the classical contributions cited above (Rubinstein, 1973; and Magni, 2007a).

Table 2: Equilibrium and Disequilibrium Net Values		
	Equilibrium	**Disequilibrium**
Net Present Value	$\frac{\overline{F}_Z}{(1+r_Z^e)} - I_Z$ $\frac{I_Z(\overline{r}_Z - r_Z^e)}{1+r_Z^e}$	$\frac{\overline{F}_Z}{(1+r_Z^d)} - I_Z$ $\frac{I_Z(\overline{r}_Z - r_Z^d)}{1+r_Z^d}$
Net Future Value (Excess Return)	$\overline{F}_Z - I_Z(1+r_Z^e)$ $I_Z(\overline{r}_Z - r_Z^e)$	$\overline{F}_Z - I_Z(1+r_Z^d)$ $I_Z(\overline{r}_Z - r_Z^d)$

Using Equations (11) to (13), Table 2 presents the various ways of representing NPVs and NFVs, in either equilibrium or disequilibrium format, which are equivalent to those presented in Definitions 1-4 above.[1]

The following section shows that the proliferation of measures under uncertainty, while surprising, is harmless in accept-reject decisions, for all of them are validly deducted from the CAPM and the assumption of share price maximization.

The Four Decision Criteria

This section shows that the four indexes introduced above are logically equivalent as decision rules in accept-reject situations. To begin with, we have the following:

Lemma 1: *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs I_Z and generates the end-of-period payoff F_Z. Then, after acceptance of the project,*

$$\overline{F}_Z - R_f\, I_Z - \lambda \operatorname{cov}\left(F_Z, r_m\right) = R_f\, N_l \left(P_l^{\circ} - P_l\right) \qquad \text{...(14)}$$

Proof: Consider firm l. Before acceptance of the project, we have, due to the Security Market Line (SML),

$$\overline{r}_l = r_f + \lambda \operatorname{cov}(r_l, r_m)$$

Reminding that $1+\overline{r}_l = \overline{F}_l / V_l$, we have,

$$\frac{\overline{F}_l}{V_l} = R_f + \lambda \operatorname{cov}\left(r_l, r_m\right)$$

and, multiplying by the firm value V_l, we obtain:

[1] It is worth reminding that if the project lies on the SML, then $I_Z = V_Z^d = V_Z^e$ and $\overline{r}_Z = r_Z^d = r_Z^e$, i.e., the three notions of rate of return collapse into one.

$$\overline{F}_l = R_f\ V_l + \lambda \operatorname{cov}(F_l, r_m) = R_f\ N_l\ P_l + \lambda \operatorname{cov}(F_l, r_m) \qquad \text{...(15)}$$

After acceptance of the project, the new equilibrium value is set as:

$$V_l^{\circ} = \frac{\overline{F}_l + \overline{F}_Z - \lambda \operatorname{cov}(F_l + F_Z, r_m)}{R_f}$$

The existing shares are N_l, so the new resulting price P_l° is such that $V_l^{\circ} - I_Z = N_l\ P_l^{\circ}$, which determines $P_l^{\circ} = \frac{V_l^{\circ} - I_Z}{N_l}$. To actually make the investment, the firm shall issue $N_l^{\circ} = \frac{I_Z}{P_l^{\circ}}$ shares at the price P_l°. The SML is now such that:

$$\frac{\overline{F}_l + \overline{F}_Z}{V_l^{\circ}} = R_f + \lambda \operatorname{cov}\left(\frac{F_l + F_Z}{V_l^{\circ}}, r_m\right)$$

whence,

$$\overline{F}_l + \overline{F}_Z = R_f\ V_l^{\circ} + \lambda \operatorname{cov}(F_l + F_Z, r_m)$$

Having determined the new price P_l° and the number N_l° of stocks issued, the above equation boils down to:

$$\overline{F}_l + \overline{F}_Z = R_f\left(N_l + N_l^{\circ}\right)P_l^{\circ} + \lambda \operatorname{cov}(F_l + F_Z, r_m) \qquad \text{...(16)}$$

Subtracting Equation (16) from Equation (15) we obtain:

$$-\overline{F}_Z = R_f\ N_l\ P_l + \lambda \operatorname{cov}(F_l, r_m) - R_f\left(N_l + N_l^{\circ}\right)P_l^{\circ} - \lambda \operatorname{cov}(F_l + F_Z, r_m)$$

and, using $N_l^{\circ} P_l^{\circ} = I_Z$,

$$\overline{F}_Z - R_f\ I_Z - \lambda \operatorname{cov}(F_Z, r_m) = R_f\ N_l\left(P_l^{\circ} - P_l\right)$$

Q.E.D.

From *Lemma 1*, four decision rules are deducted. We first prove the legitimacy of the disequilibrium NPV (this proposition is just Magni's, 2009 Proposition 1).

> ***Proposition 1:*** *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs* I_Z *and generates the end-of-period payoff* F_Z. *The firm's share price increases if and only if the project disequilibrium NPV is positive:*[2]

[2] It is assumed that R_f and $R_f + (\lambda/I_Z) \operatorname{cov}(F_Z, r_m)$ have same sign. If this condition is not met, the thesis holds with the sign of Equation (17) reversed.

$$dNPV_Z := \frac{\overline{F}_Z}{R_f + \frac{\lambda}{I_Z}\,\text{cov}\left(F_Z, r_m\right)} - I_Z > 0 \qquad \text{...(17)}$$

Proof: From Equation (14) we find,

$$\overline{F}_Z - I_Z\left[R_f + \lambda\,\text{cov}\left(\frac{F_Z}{I_Z}, r_m\right)\right] = R_f\;N_l\left(P_l^{\circ} - P_l\right)$$

whence,

$$\frac{\overline{F}_Z}{R_f + \lambda\,\text{cov}\left(\frac{F_Z}{I_Z}, r_m\right)} - I_Z = \frac{R_f\;N_l\left(P_l^{\circ} - P_l\right)}{R_f + \lambda\,\text{cov}\left(\frac{F_Z}{I_Z}, r_m\right)}$$

Therefore,

$P_l^{\circ} > P_l$ if and only if $V_Z^d - I_Z = dNPV_Z > 0$

Q.E.D.

> ***Proposition 2:*** *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs I_Z and generates the end-of-period payoff F_Z. The firm's share price increases if and only if the project equilibrium NPV is positive:*

$$eNPV_Z := \frac{\overline{F}_Z}{R_f + \frac{\lambda}{V_Z^e}\,\text{cov}\left(F_Z, r_m\right)} - I_Z > 0 \qquad \text{...(18)}$$

Proof: Using Equation (14) and the fact that $\overline{F}_Z - \lambda\,\text{cov}\left(F_Z, r_m\right) = R_f\,V_Z^e$, we have:

$$R_f\,V_Z^e - R_f\,I_Z = R_f\;N_l\left(P_l^{\circ} - P_l\right)$$

whence dividing by R_f, we get:

$$eNPV_Z = N_l\left(P_l^{\circ} - P_l\right) \qquad \text{...(19)}$$

Finally, we have:

$P_l^{\circ} > P_l$ if and only if $eNPV_Z > 0$

Q.E.D.

> ***Proposition 3:*** *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs I_Z and generates the end-of-period payoff F_Z. The firm's share price increases if and only if the project disequilibrium*

NFV is positive:

$$dNFV_Z = I_Z \left(\bar{r}_Z - r_Z^d\right) > 0 \qquad ...(20)$$

Proof: From Equation (14) we have:

$$\bar{F}_Z - I_Z \left(R_f + \lambda \operatorname{cov}\left(r_Z, r_m\right)\right) = R_f \; N_l \left(P_l^\circ - P_l\right) \qquad ...(21)$$

Given that,

$$dNFV_Z = I_Z \left(\bar{r}_Z - r_Z^d\right) = \bar{F}_Z - I_Z \left(R_f + \lambda \operatorname{cov}\left(r_Z, r_m\right)\right) \qquad ...(22)$$

we have:

$P_l^\circ > P_l$ if and only if $dNFV_Z > 0$

Q.E.D.

Proposition 4: *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking a project Z that costs I_Z and generates the end-of-period payoff F_Z. The firm's share price increases if and only if the project equilibrium NFV is positive:*[3]

$$eNFV_Z = I_Z \left(\bar{r}_Z - r_Z^e\right) > 0 \qquad ...(23)$$

Proof: Using Equation (14) and the equalities $\bar{F}_Z - \lambda \operatorname{cov}\left(F_Z, r_m\right) = R_f \, V_Z^e = R_f \, \bar{F}_Z \big/ \left(1 + r_Z^e\right)$, we have

$$R_f \frac{\bar{F}_Z}{(1 + r_Z^e)} - R_f \, I_Z = R_f \; N_l \left(P_l^\circ - P_l\right)$$

and therefore,

$$\bar{F}_Z \, R_f - R_f \left(1 + r_Z^e\right) I_Z = \left(1 + r_Z^e\right) R_f \; N_l \left(P_l^\circ - P_l\right)$$

whence dividing by R_f,

$$\bar{F}_Z - I_Z \left(1 + r_Z^e\right) = \left(1 + r_Z^e\right) N_l \left(P_l^\circ - P_l\right)$$

which leads to:

$P_l^\circ > P_l$ if and only if $eNFV_Z > 0$

Q.E.D.

Remark 2: Propositions 1-4 show four ways of using the CAPM for capital budgeting purposes. All of them are CAPM-consistent. In particular, it is worth stressing that: (a) The disequilibrium NPV is indeed a correct decision rule, despite some claims against its use

[3] Here it has been assumed that $\left(1 + r_Z^e\right) > 0$. If this condition is not met, then the thesis holds with the sign of Equation (23) reversed.

(e.g., De Reyck, 2005); (b) The NPV rule may be safely replaced by a NFV (excess return) rule, either in equilibrium or disequilibrium format.

Remark 3: The results obtained have some practical consequences. In real life, investors face several different situations in capital budgeting. In particular, information about the project may be extensive or partial so that project analysis may or may not rely on a scenario basis, and there may or may not be assets in the security market having economic characteristics similar to those of the project under consideration (representative assets). If appropriate information on the project is available (so that scenario analysis is possible) and/or there are no representative assets in the market, the investor must rely on an *ex ante* probability distribution to compute the covariance between the end-of-period cash flow and the market return, $\text{cov}(F_Z, r_m)/I_Z$. This means that the investor will equivalently employ the disequilibrium NPV or the disequilibrium NFV to decide whether to invest or not in the project. If appropriate information is somehow lacking and there are representative assets in the security market, the decision maker may measure the covariance from the historical return data of representative assets. The covariance so obtained, is a proxy for the equilibrium covariance, $\text{cov}\left(r_Z^e, r_m\right) = \text{cov}\left(F_Z, r_m\right)/V_Z^e$ (assuming the market is in equilibrium)[4] and the investor will therefore employ the equilibrium NPV or the equilibrium NFV. In both cases, the decision maker is reliably supported by a pair of metrics that lead to correct decisions.

Nonadditivity

This section shows that the disequilibrium NFV and the equilibrium NPV are additive, whereas the disequilibrium NPV and the equilibrium NFV are nonadditive. NPV additivity implies:

$$NPV_{Z_1} + NPV_{Z_2} = NPV_{Z_1+Z_2} \quad \text{for any pair of projects } Z_1, Z_2 \qquad \text{...(24)}$$

(analogously for the NFV). Therefore, to show nonadditivity it suffices to provide a counterexample, i.e., a pair of projects (or a class of pairs of projects) for which Equation (24) does not hold. We first begin with the disequilibrium NPV. The following proposition is just Magni's (2009) Proposition 9, but the proof is different.

Proposition 5: *The disequilibrium NPV is nonadditive.*

Proof: Consider a pair of projects Z_1 and Z_2 such that $Z_1 = (-h, k)$ and $Z_2 = (-I_Z + h, F_Z - k)$ with h, k being any nonzero real numbers (note that $Z = Z_1 + Z_2$). Consider the function

$$f(h,k) := \overbrace{\left(-(I_Z - h) + \frac{\overline{F}_Z - k}{R_f + \dfrac{\lambda\,\text{cov}\left(F_Z, r_m\right)}{\left(I_Z - h\right)}}\right)}^{dNPV_{Z_1}} + \overbrace{\left(-h + \frac{k}{R_f}\right)}^{dNPV_{Z_2}}$$

[4] If the market is not in equilibrium, the historical covariances are not proxies for the equilibrium covariances and one must rely on the previous method (disequilibrium covariance). However, in this case, one should actually wonder whether the CAPM should be applied, given that equilibrium is a fundamental assumption of the model. This issue is an important practical problem, but is beyond the scope of this paper.

If the disequilibrium NPV were additive, then Equation (24) would hold and $f(h, k)$ would be constant under changes in h and k (in particular, we would have $f(h, k) = f(0, 0) = dNPV_Z$ for all h, k). But,

$$\frac{\partial f(h,k)}{\partial h} = -\frac{\lambda \operatorname{cov}(F_Z, r_m)(\overline{F}_Z - k)}{\left[R_f(I_Z - h) + \lambda \operatorname{cov}(F_Z, r_m)\right]^2}$$

$$\frac{\partial f(h,k)}{\partial k} = \frac{1}{R_f} - \frac{1}{R_f + \dfrac{\lambda \operatorname{cov}(F_Z, r_m)}{(I_Z - h)}}$$

which, in general, are not identically zero. Therefore $f(h, k)$ is not invariant with respect to h and k.

Q.E.D.

Proposition 6: *The equilibrium NPV is additive.*

Proof: Consider any pair of projects Z_1 and Z_2, with I_{Z_1} and I_{Z_2} being the respective outlays, while F_{Z_1} and F_{Z_2} are the respective end-of-period outcomes. Let $I_Z := I_{Z_1} + I_{Z_2}$ and $F_Z := F_{Z_1} + F_{Z_2}$. Using the certainty-equivalent form of the equilibrium NPV (see Equation 7), we have:

$$eNPV_{Z_1} + eNPV_{Z_2} = \frac{\overline{F}_{Z_1} - \lambda \operatorname{cov}(F_{Z_1}, r_m)}{R_f} - I_{Z_1} + \frac{\overline{F}_{Z_2} - \lambda \operatorname{cov}(F_{Z_2}, r_m)}{R_f} - I_{Z_2}$$

$$= \frac{\overline{F}_Z - \lambda \operatorname{cov}(F_Z, r_m)}{R_f} - I_Z = eNPV_Z$$

Q.E.D.

Proposition 7: *The disequilibrium NFV is additive.*

Proof: Reminding that $dNFV_Z = \overline{F}_Z - I_Z(1 + r_Z^d)$ (see Table 2) we have:

$$dNFV_{Z_1} + dNFV_{Z_2} = \left(\overline{F}_{Z_1} - R_f I_{Z_1} - \frac{\lambda}{I_{Z_1}} \operatorname{cov}(F_{Z_1}, r_m) I_{Z_1}\right) + \left(\overline{F}_{Z_2} - R_f I_{Z_2} - \frac{\lambda}{I_{Z_2}} \operatorname{cov}(F_{Z_2}, r_m) I_{Z_2}\right)$$

$$= (\overline{F}_Z - I_Z) - r_f I_Z - \lambda \operatorname{cov}(F_Z, r_m)$$

$$= dNFV_Z$$

Q.E.D.

Proposition 8: *The equilibrium NFV is nonadditive.*

Proof: Consider a pair of projects Z_1 and Z_2 such that $Z_1 = (-h, k)$ and $Z_2 = (-I_Z + h, F_Z - k)$ with h, k being any nonzero real numbers (note that $Z = Z_1 + Z_2$). Taking into consideration Equation (10) and reminding that $\text{cov}(k, r_m) = 0$ for all $k \in R$, we consider the function:

$$g(h,k) = \overbrace{\left(\left(\overline{F}_Z - k\right) - \left(I_Z - h\right) - \left(r_f + \frac{\lambda \text{ cov}\left(F_Z - k, r_m\right)}{V^e_{Z_1}} \right) \left(I_Z - h\right) \right)}^{eNFV_{Z_1}} + \overbrace{\left(k - h - r_f\, h\right)}^{eNFV_{Z_2}}$$

Manipulating algebraically, we get:

$$g(h,k) = \overline{F}_Z - R_f\, I_Z - \frac{\lambda \text{cov}\left(F_Z, r_m\right)\left(I_Z - h\right)}{V^e_{Z_1}}$$

with

$$V^e_{Z_1} = V^e_{Z_1}(k) = \frac{\overline{F}_Z - k - \lambda \text{cov}\left(F_Z - k, r_m\right)}{R_f}$$

so that

$$\frac{\partial g(h,k)}{\partial h} = \frac{\lambda \text{ cov}\left(F_Z, r_m\right)}{V^e_{Z_1}(k)}$$

$$\frac{\partial g(h,k)}{\partial k} = -\frac{\lambda \text{ cov}\left(F_Z, r_m\right)\left(I - h\right)}{R_f \left[V^e_{Z_1}(k)\right]^2}$$

which, in general, are not identically zero.

Q.E.D.

Table 3 summarizes the results obtained, showing that additivity is, so to say, two-dimensional, depending on the two pairs equilibrium/disequilibrium and present/future.

Table 3: Additive and Nonadditive Net Values

	Equilibrium	**Disequilibrium**
Net Present Value	Additive	Nonadditive
Net Future Value	Nonadditive	Additive

Table 4 illustrates a numerical example where a decision maker is supposed to be evaluating two risky projects. The security market is composed, for the sake of simplicity, of a single risky security (so that its rate of return coincides with the market rate of return, r_m). One of the three states of nature may occur with probabilities equal to 0.4, 0.3 and 0.4 respectively. The risk-free security has a face value of 120 and a price of 90. The risk-free rate is therefore 33.33% (=120/90 – 1). To compute the four net values, we use Equation (5) (dNPV) and Equation (7) (eNPV), while the dNFV (Equation 8) and the eNFV (Equation 9) are found by multiplying the former by $(1+r_Z^d)$ and the latter by $(1+r_Z^e)$ (Equations 12 and 13). Consistent with the above-mentioned propositions, the sum of the dNPVs (eNFVs) of the two

Table 4: Project Valuation with the CAPM

	Proj. Z_1	**Proj. Z_2**	**Proj. $Z_1 + Z_2$**	**Proj. Z_1 + Proj. Z_2**	**Risky Security**	**Risk-Free Security**	**Market (3,000 sh.)**
F	$\begin{cases}130\\100\\10\end{cases}$	$\begin{cases}120\\50\\80\end{cases}$	$\begin{cases}250\\150\\90\end{cases}$	–	$\begin{cases}98\\71\\100\end{cases}$	$\begin{cases}120\\120\\120\end{cases}$	$\begin{cases}294{,}000\\213{,}000\\300{,}000\end{cases}$
Cost/Price	70	30	100	–	54	90	162,000
Rate of Return (%)	$\begin{cases}85.7\\42.8\\-85.7\end{cases}$	$\begin{cases}300\\66.6\\166.6\end{cases}$	$\begin{cases}150\\50\\-10\end{cases}$	–	$\begin{cases}81.48\\31.48\\85.18\end{cases}$	$\begin{cases}33.3\\33.3\\33.3\end{cases}$	$\begin{cases}81.48\\31.48\\85.18\end{cases}$
$\overline{F}$	82.00	80.00	162.00	–	87.80	120.00	263,400
$\bar{r}$ (%)	17.14	166.66	62.00	–	62.59	33.33	62.59
λ	4.52	–	–	–	–	–	–
λ cov(F, r_m)	–19.88	27.12	7.23	–	15.80	0	47,400
$\overline{F} - \lambda \text{ cov}(F, r_m)$	82.00	80.00	162.00	–	87.80	120.00	263,400
Disequilibrium Value	78.15	35.75	115.25	113.90	54.00	90.00	162,000
Equilibrium Value	76.41	39.66	116.07	116.07	54.00	90.00	162,000
Disequilibrium NPV	8.15	5.75	15.24	13.90	0	0	0
Equilibrium NPV	**6.41**	**9.66**	16.07	16.07	0	0	0
Disequilibrium NFV	**8.55**	**12.88**	21.43	21.43	0	0	0
Equilibrium NFV	6.88	19.49	22.44	26.37	0	0	0

projects is not equal to the dNPV (eNFV) of the project obtained by summing the two projects' cash flows. Conversely, the eNPV and the dNFV are additive, which confirms the economic interpretation of these indexes as valuation tools. Equation (19) represents the eNPV as the price increase times the number of shares outstanding, which exactly measures the increase in shareholders' wealth if the project is undertaken.

Remark 4: It is worth noting that the dNFV and the eNPV are risk-free-related, so to say, in the sense that the equilibrium NPV is just the discounted value of the disequilibrium NFV, where the discount rate is the risk-free rate of the security market:

$$\frac{dNFV}{R_f} = \frac{\overline{F}_Z - I_Z\left(R_f + \frac{\lambda}{I_Z}\text{ cov}\left(F_Z, r_m\right)\right)}{R_f} = \frac{\overline{F}_Z - \lambda \text{ cov}\left(F_Z, r_m\right)}{R_f} - I_Z = eNPV \qquad ...(25)$$

Referring to the example of Table 4 and, in particular, to projects Z_1 and Z_2, we get the eNPV as 6.41 (=8.55/1.3333) and 9.66 (=12.88/1.3333) respectively. This fact may be interpreted in an arbitrage perspective. Suppose a shareholder owns n shares of the firm; before acceptance of the project, the value of his portfolio is nP_l, after acceptance it becomes nP_l°. Suppose he sells m shares, with $m = n\left(P_l^\circ - P_l\right)/P_l^\circ$. Then the value of his investment in the firm becomes $nP_l^\circ - mP_l^\circ = nP_l$ as before acceptance of the project. If he invests the proceeds at the risk-free rate, he will have, at the end of the period, a certain amount equal to:

$$R_f mP_l^\circ = R_f n\left(P_l^\circ - P_l\right) = \left(\frac{n}{N_l}\right) R_f\, N_l \left(P_l^\circ - P_l\right) = \left(\frac{n}{N_l}\right) dNFV$$

where we have used Equations (19) and (25). By undoing the increase in the firm value, the investor will assure himself an arbitrage profit equal to that part of the dNFV that corresponds to his investment in the firm. To put it differently, the dNFV is the (total) arbitrage profit that the shareholders get at the end of the period if the project is undertaken.

Remark 5: The dNPV and the eNFV may only be used as decision rules.[5] However, nonadditivity has something to do with decision as well. Given an investment, Equation (17) does hold, but dealing with two investments to be both accepted or rejected (or an investment composed of two sub-investments), one may not deduce whether the portfolio of the two projects is profitable if the sum of the two NPVs is positive. In other words, before applying Equation (17), one must first consider the overall cash flows deriving from the two investments, and only afterwards compute the NPV. To calculate the NPV of each investment and then sum the NPVs is not compatible with *Proposition 1*. This boils down to saying that the disequilibrium NPV is dangerous if used for decision purposes, because decision makers coping with two or more projects (or a single project that is composed of several subprojects) may be tempted to first compute the NPV of each project and then sum the NPVs. This procedure may lead to a different sign than the one obtained with the correct procedure. It is easy to show that there may be instances where the sign of $NPV_{Z_1} + NPV_{Z_2}$ does not coincide with the sign of $NPV_{Z_1+Z_2}$. Consider again the example in Table 4 and suppose the cost of project Z_2 is equal to 48 (other things unvaried). A simple calculation shows that $dNPV_{Z_1} + dNPV_{Z_2}$ = 8.15 + (–5.86) = 2.29 > 0 while $dNPV_{Z_1+Z_2}$ = –1.84 < 0 (i.e., this portfolio of projects is profitable or not depends on how the investor computes the overall NPV).

The same remarks obviously hold for the equilibrium NFV. For example, if one sets the cost of project Z_2 at €45 (other things unvaried) we have:

$$eNFV_{Z_1} + eNFV_{Z_2} = 6.88 + (-10.77) = -3.89 < 0 \quad \text{and} \quad eNFV_{Z_1+Z_2} = 1.5 > 0$$

Remodeling the Decision Problem

Though the dNPV and the eNFV are nonadditive, they are impeccably deducted from the CAPM assumptions. One may well dismiss them by invoking additivity. Additivity is a

[5] Given that the disequilibrium NPV and the equilibrium NFV are not valuation tools, to use the term 'value' for labelling them is admittedly improper.

cardinal assumption in finance and nonadditive measures are unacceptable. However, in modelling a decision criterion, one should preferably obtain (rather than assume) additivity, i.e., one should not resort to additivity as an ad hoc assumption to get rid of unpleasant (though logically deduced) results; additivity should be a logical consequence of the criterion at hand. This section shows that the dNPV and eNFV cannot be logically derived from the CAPM assumptions, if the decision process is reframed in a more general way.

First, note that *Lemma 1* is based on a well-determined problem:

An economic agent faces the opportunity of investing in project *Z*.

Should the decision maker invest in *Z* or not? (DP-1)

The dichotomy is: Undertake *Z*/do not undertake *Z*. Formally, the two alternatives are described by the equilibrium Relations (16) and (15) respectively, which we rewrite below for the benefit of the reader:

Z is undertaken $\rightarrow$ $\overline{F}_l + \overline{F}_Z = R_f \left(N_l + N_l^{\circ}\right) P_l^{\circ} + \lambda \operatorname{cov}\left(F_l + F_Z, r_m\right)$

Z is not undertaken $\rightarrow$ $\overline{F}_l = R_f V_l + \lambda \operatorname{cov}\left(F_l, r_m\right) = R_f N_l P_l + \lambda \operatorname{cov}\left(F_l, r_m\right)$

The difference between the two equations leads to Equation (14), which logically implies Propositions 1 and 4 (which in turn legitimize the use of the dNPV and the eNFV for decision making). Let us now change the framing of the problem into the following:

An economic agent faces the opportunity of investing in project *Z* or in project *Y*.

Should the decision maker invest in *Z* or in *Y*? (DP-2)

The Decision Problem (DP-2) states that the decision maker faces two alternatives, named ‘project *Z*’ and ‘project *Y*’. The problem (DP-2) is a generalization of (DP-1); the latter may be obtained from the former by stating that ‘project *Y*’ is the null alternative, that is, a project with zero cash flows. It is just this general framing which prevents the dNPV and eNFV to be deducted from the CAPM assumptions.

Proposition 9: *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking either project Z, which costs* I_Z *and generates the end-of-period payoff* F_Z, *or project Y, which costs* I_Y *and generates the end-of-period payoff* F_Y. *The firm’s share price increases if and only if project Z’s eNPV (respectively, dNFV) is greater than project Y’s eNPV (respectively, dNFV).*

Proof: If *Z* is undertaken, the equilibrium relation will be:

Z is undertaken $\rightarrow$ $\overline{F}_l + \overline{F}_Z = R_f \left(N_l + N_l^{\circ}\right) P_l^{\circ} + \lambda \operatorname{cov}\left(F_l + F_Z, r_m\right)$ ...(16)

If *Y* is undertaken, an analogous equilibrium relation will hold, where *Y* replaces *Z*:

Y is undertaken $\rightarrow$ $\overline{F}_l + \overline{F}_Y = R_f \left(N_l + N_l^{\bullet}\right) P_l^{\bullet} + \lambda \operatorname{cov}\left(F_l + F_Y, r_m\right)$ ...(16-bis)

Subtracting Relation (16-bis) from Relation (16), we get:

$$\left[\overline{F}_Z - \lambda \operatorname{cov}(F_Z, r_m) - R_f I_Z\right] - \left[\overline{F}_Y - \lambda \operatorname{cov}(F_Y, r_m) - R_f I_Y\right] = R_f N_l \left(P_l^{\circ} - P_l^{\bullet}\right)$$

where we use the equality $N_l^{\bullet} P_l^{\bullet} = I_Y$. Therefore,

$$R_f \left[\left(V_Z^e - I_Z\right) - \left(V_Y^e - I_Y\right)\right] = R_f N_l \left(P_l^{\circ} - P_l^{\bullet}\right) \qquad ...(26)$$

so that,

$$P_l^{\circ} > P_l^{\bullet}, \text{ if and only if } eNPV_Z > eNPV_Y \qquad ...(27a)$$

thus, proving the first part of the proposition. Owing to Equation (25), we also have:

$$P_l^{\circ} > P_l^{\bullet}, \text{ if and only if } dNFV_Z > dNFV_Y \qquad ...(27b)$$

(as long as $R_f > 0$), thus proving the second part.

Q.E.D.

Proposition 9 tells us that the decision rule deducted from the CAPM assumptions and (DP-2) is:

Invest in *Z* if and only if its eNPV (dNFV) is greater than *Y*'s eNPV (dNFV).

Corollary 1: *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking project Z, which costs I_Z and generates the end-of-period payoff F_Z. The firm's share price increases if and only if project Z's eNPV (respectively dNFV) is positive.*

Proof: The assumptions are the same as in *Proposition 9*, with *Y* being the null alternative (with cash flows equal to zero). Then, the NPV in Equation (27a) (the net finale value in Equation 27b) is zero, and the criterion becomes:

Invest in *Z* if and only if the eNPV (dNFV) is positive.

Q.E.D.

We now prove that Equations (17) and (23) cannot be deduced from (DP-2).

Proposition 10: *Suppose all CAPM assumptions are met, and a firm l has the opportunity of undertaking either project Z, which costs I_Z and generates the end-of-period payoff F_Z or project Y, which costs I_Y and generates the end-of-period payoff F_Y. The dNPV rule and the eNFV rule cannot be derived.*

Proof: As seen, problem (DP-2) implies Equations (16) and (16-bis). If the dNPV rule is deductible from these equations, then it must be:

$$P_l^{\circ} > P_l^{\bullet}, \text{ if and only if } \frac{\overline{F}_Z}{R_f + \lambda \operatorname{cov}\left(\frac{F_Z}{I_Z}, r_m\right)} - I_Z > \frac{\overline{F}_Y}{R_f + \lambda \operatorname{cov}\left(\frac{F_Y}{I_Y}, r_m\right)} - I_Y \qquad ...(28)$$

However, this excludes the case where Y is the null alternative, because I_Y cannot be zero. Furthermore, subtracting Equation (16-bis) from Equation (16) and manipulating, we get:

$$\overline{F}_Z - \overline{F}_Y - (I_Z - I_Y)\left(R_f + \lambda \operatorname{cov}\left(\frac{F_Z - F_Y}{I_Z - I_Y}, r_m\right)\right) = N_l R_f \left(P_l^{\circ} - P_l^{\bullet}\right)$$

whence,

$$P_l^{\circ} > P_l^{\bullet}, \text{ if and only if } \frac{\overline{F}_Z}{R_f + \lambda \operatorname{cov}\left(\frac{F_Z - F_Y}{I_Z - I_Y}, r_m\right)} - I_Z > \frac{\overline{F}_Y}{R_f + \lambda \operatorname{cov}\left(\frac{F_Z - F_Y}{I_Z - I_Y}, r_m\right)} - I_Y \quad \text{...(29)}$$

which is not equivalent to Equation (28).

As for the eNFV rule to be valid, it must be,

$$P_l^{\circ} > P_l^{\bullet} \text{ if and only if } \left(\overline{F}_Z - I_Z\right) - r_Z^e I_Z > \left(\overline{F}_Y - I_Y\right) - r_Y^e I_Y \quad \text{...(30)}$$

But subtracting Equation (16-bis) from Equation (16) and using the equalities, $\overline{F}_j - \lambda \operatorname{cov}\left(F_j, r_m\right) = V_j^e R_f$ and $1 + r_j^e = \dfrac{\overline{F}_j}{V_j^e}$, j = Z, Y, algebraic manipulations lead to:

$$P_l^{\circ} > P_l^{\bullet}, \text{ if and only if } \frac{\left(\overline{F}_Z - I_Z\right) - r_Z^e I_Z}{1 + r_Z^e} > \frac{\left(\overline{F}_Y - I_Y\right) - r_Y^e I_Y}{1 + r_Y^e} \quad \text{...(31)}$$

which is not equivalent to Equation (30).

Q.E.D.

The eNFV and dNPV rule are thus removed with no need of invoking additivity. They are simply, not deductible from the CAPM assumptions if the decision problem is (DP-2), which transforms (and generalizes) the dichotomy 'undertake *Z*/do not undertake *Z*' into 'undertake *Z*/undertake *Y*'.

Decreasing Net Values and Project Valuation

The previous sections have shown that only eNPV and dNFV can be legitimately deducted from the CAPM and an appropriate decision problem. They are deducted not only as decision rules but also as valuation tools. In other words, they provide the project value (current and future respectively). This section shows that, despite their additivity, the eNPV or dNFV may be misleading in some cases.

Consider a project whose random end-of-period payoff is $F_Z^k \in R$, if state k occurs, k = 1, 2, …, n. The project disequilibrium NFV and the project equilibrium NPV may be represented as functions of n variables:

$$dNFV\left(F_Z^1, F_Z^2, ..., F_Z^n\right) = \overline{F}_Z - I_Z\left(R_f + \frac{\lambda}{I_Z} \operatorname{cov}\left(F_Z, r_m\right)\right)$$

$$= \sum_{k=1}^{n} p_k F_Z^k - I_Z R_f - \lambda \left(\sum_{k=1}^{n} p_k F_Z^k r_{mk} - \bar{r}_m \sum_{k=1}^{n} p_k F_Z^k \right) \qquad ...(32)$$

$$eNFV\left(F_Z^1, F_Z^2, ..., F_Z^n\right) = \frac{\bar{F}_Z - \lambda \operatorname{cov}\left(F_Z, r_m\right)}{R_f} - I_Z$$

$$= \frac{\sum_{k=1}^{n} p_k F_Z^k - \lambda \left(\sum_{k=1}^{n} p_k F_Z^k r_{mk} - \bar{r}_m \sum_{k=1}^{n} p_k F_Z^k \right)}{R_f} - I_Z \qquad ...(33)$$

where p_k is the probability of state k. For functions (32) and (33) to provide correct (net) values, they must abide by the no-arbitrage principle. In other words, increasing end-of-period cash flows should lead to increasing values, *ceteris paribus*. Consider two assets Z and W that may be purchased at the same price. Suppose $F_Z^k = F_W^k$ for all k but s, with $F_Z^s < F_W^s$. Asset W may then be seen as asset Z plus an arbitrage profit paying off nonnegative amounts in all states and a strictly positive amount $\left(F_W^s - F_Z^s\right)$ in state s. Asset W's value must therefore be higher than asset Z's, otherwise arbitrage opportunities arise.[6] From a capital budgeting perspective, given a determined eNPV and dNFV for project Z, project W must have higher eNPV and dNFV (assuming their costs are equal), which boils down to $\frac{\partial}{\partial F_Z^k} dNFV > 0$ and $\frac{\partial}{\partial F_Z^k} eNPV > 0$ for every k = 1, 2, …, n. If, instead, the project under consideration is such that:

$$\frac{\partial}{\partial F_Z^s} dNFV < 0 \quad \text{and} \quad \frac{\partial}{\partial F_Z^s} eNPV < 0 \quad \text{for some } s \qquad ...(34)$$

the dNFV and the eNPV do not provide a reliable valuation, because they are inconsistent with the no-arbitrage principle. From Equation (32) we have that:

$$\frac{\partial}{\partial F_Z^s} dNFV = \frac{\partial}{\partial F_Z^s} \left[\sum_{k=1}^{n} p_k F_Z^k - I_Z R_f - \lambda \left(\sum_{k=1}^{n} p_k F_Z^k r_{mk} - \bar{r}_m \sum_{k=1}^{n} p_k F_Z^k \right) \right] \qquad ...(35)$$

and, owing to Equation (25),

$$\frac{\partial}{\partial F_Z^s} eNPV = \frac{1}{R_f} \left(\frac{\partial}{\partial F_Z^s} dNFV \right) \qquad ...(36)$$

[6] From the stochastic dominance perspective, note that asset *W* dominates *Z* according to both first-order and second-order stochastic dominance.

Also, it is evident that:

$$\frac{\partial}{\partial F_Z^s}\sum_{k=1}^{n} p_k F_Z^k = p_s \qquad ...(37)$$

$$\frac{\partial}{\partial F_Z^s}\left[\sum_{k=1}^{n} p_k F_Z^k r_{mk} - \bar{r}_m \sum_{k=1}^{n} p_k F_Z^k\right] = p_s r_{ms} - \bar{r}_m p_s \qquad ...(38)$$

with r_{mk} being the market rate of return if state k occurs. Therefore, we may write:

$$\frac{\partial}{\partial F_Z^s} dNFV = p_s - \lambda p_s r_{ms} + \bar{r}_m \lambda p_s \text{ and } \frac{\partial}{\partial F_Z^s} eNPV = \left(p_s - \lambda p_s r_{ms} + \bar{r}_m \lambda p_s\right)/R_f \qquad ...(39)$$

Let us now consider project Z in Table 5. Considering its dNFV and eNPV as functions of F_Z^3 (end-of-period cash flow if state 3 occurs) and using Equation (39), we find that Condition (34) is satisfied for s = 3:

$$\frac{\partial}{\partial F_Z^3} dNFV = 0.3 - 4.52\,(0.3)(0.8518 - 0.6259) < 0$$

Table 5: Decreasing Net Values

	Project Z	**Risky Security**	**Risk-Free Security**	**Market (000,000) (3 million shares)**	**Probability**
F	$\begin{cases}\mathbf{98}\\ \mathbf{71}\\ \mathbf{350}\end{cases}$	$\begin{cases}\mathbf{98}\\ \mathbf{71}\\ \mathbf{100}\end{cases}$	$\begin{cases}120\\ 120\\ 120\end{cases}$	$\begin{cases}294\\ 213\\ 300\end{cases}$	$\begin{cases}0.3\\ 0.4\\ 0.3\end{cases}$
Cost/Price	**54**	**54**	90	162	
r(%)	$\begin{cases}81.48\\ 31.48\\ 548.1\end{cases}$	$\begin{cases}81.48\\ 31.48\\ 85.18\end{cases}$	$\begin{cases}33.3\\ 33.3\\ 33.3\end{cases}$	$\begin{cases}81.48\\ 31.48\\ 85.18\end{cases}$	$\begin{cases}0.3\\ 0.4\\ 0.3\end{cases}$
$\bar{r}$ (%)	201.48	62.59	33.33	62.59	
$\bar{F}$	162.8	87.8	120	263.4	
cov(F, r_m)	20.44	3.49	0	10.486	
λ	4.52				
λ cov(F, r_m)	92.39	15.8	0	47.4	
$\bar{F}$ – λ cov(F, r_m)	70.41	72	120	216	
V^e	**52.808**	**54**	90	162	
Equilibrium NPV	**–1.19**	**0**	0	0	
Disequilibrium NFV	**–1.59**	**0**	0	0	

$$\frac{\partial}{\partial F_Z^3} eNPV = \frac{1}{1.3333}\left[0.3 - 4.52\,(0.3)\,(0.8518 - 0.6259)\right] < 0$$

This means 'the more the payoff, the less the value', which is incompatible with an arbitrage-free evaluation. Note that project *Z* may be seen as the risky security plus an arbitrage profit that pays off nonnegative cash flows in all states and a strictly positive amount of 250 if state 3 occurs.[7] Therefore, project *Z* must have a higher (net) value than the risky security. Given that the net values of the risky security are zero (for the risky security lies on the SML), project *Z*'s net values must be positive. Both first-order and second-order stochastic dominance confirm the natural intuition according to which *Z* dominates the risky security. Yet, both the equilibrium NPV and the disequilibrium NFV are negative. They signal non-profitability for project *Z* (the equilibrium value is 52.808, smaller than the cost) or, equivalently, they do not signal that the project gives the investor an arbitrage opportunity. This implies that, the dNPV and eNFV are not additive, but the eNPV and dNFV though additive, have pitfalls as well.

This enables us to state the following:

Proposition 11: *Suppose that*

a. The security market is in equilibrium

b. Condition (34) holds, i.e., $\frac{\partial}{\partial F_Z^s} dNFV < 0$ *and* $\frac{\partial}{\partial F_Z^s} eNPV < 0$ *for some s*

Then, the eNPV and the dNFV may not be used for valuation (nor decision) purposes.

Proposition 11 is related to a previous result found by Dybvig and Ingersoll (1982, p. 237). The authors, dealing with pricing of marketed assets in a complete market, prove the following:

Dybivg and Ingersoll's Proposition (DIP): *Suppose that*

1. *Mean-variance pricing holds for all assets, i.e.,* $\bar{r}_l - r_f = \lambda \operatorname{cov}(r_l, r_m)$ *with* $r_f, \lambda > 0$
2. *Markets are complete so that any payoff across states can be purchased as some portfolio of marketed securities; and*
3. *The market portfolio generates sufficiently large returns in some state(s), i.e.,* $\text{prob}\ (r_m > \bar{r}_m + 1/\lambda) > 0$

Then there exists an arbitrage opportunity.

Remark 6: It is worth noting that condition (b) of *Proposition 11* is equivalent to Dybvig and Ingersoll's condition (3), because $\text{prob}\ (r_m > \bar{r}_m + 1/\lambda) > 0$, if and only if $r_{ms} > \bar{r}_m + 1/\lambda$ for some *s*, which means $\lambda(r_{ms} - \bar{r}_m) > 1$ for some *s*, and, owing to Equation (39) and the fact that

[7] It is possible to set the project's cost lower than the risky security's price, so that the arbitrage becomes a strong arbitrage, with a positive net cash flow at time 0 and nonnegative amount (possibly positive) at time 1.

$p_s > 0$ and $R_f > 0$, the latter holds if and only if $\frac{\partial}{\partial F_Z^s} dNFV < 0$ and $\frac{\partial}{\partial F_Z^s} eNPV < 0$ for some s.[8]

As a result, the two assumptions (a) and (b) in *Proposition 11* imply that the market is not complete. To understand why, consider that if the market was complete and condition (b) holds, then condition (2) and (3) of *DIP* would also hold. But then the market would not be in equilibrium, as arbitrage opportunities arise (see Dybvig and Ingersoll, 1982, p. 238). Therefore, assumptions (a) and (b) are only compatible with an incomplete market.

The result presented in *Proposition 11* is, so to say, the capital-budgeting counterpart of *DIP*. In particular, while the latter deals with pricing of marketed assets when the security market is complete, the former deals with valuation of non-marketed assets (projects) when the security market is incomplete. The two propositions are the two sides of the same coin and the two perspectives are perfectly reconciled (see Table 6).

Table 6: Range of Applicability of *DIP* and *Proposition 11*

	Security Market	**Type of Assets**
DIP	Complete	Securities (Marketed assets)
Proposition 11	Incomplete	Projects (Non-marketed assets)

Equilibrium Value and Counterfactual Equilibrium Price

This section shows that the equilibrium value of a project is not necessarily the value the project would have if it were traded.

We now consider Equation (7). It says that the eNPV is just the difference between the equilibrium value and the cost of the project: $eNPV_Z = V_Z^e - I_Z$ where

$$V_Z^e = \frac{\bar{F}_Z - \lambda \mathrm{cov}(F_Z, r_m)}{R_f} \qquad \text{...(40)}$$

In finance, V_Z^e is known as the 'equilibrium value' of the project. It is commonly believed that it is the price that the project would have in equilibrium if it was traded in the security market (e.g., Mason and Merton, 1985, pp. 38-39; and Smith and Nau, 1995, p. 800). But this equivalence does not always hold good, as Smith and Nau (1995) clearly point out:

> We also have some semantic problems defining exactly what is meant by the value of a non-traded project. Earlier the … value of a project was defined as the price the project would have if it was traded in an arbitrage-free market…. This definition does not work well in general because the introduction of the project into the market may create new investment opportunities and change the prices of the traded securities (Smith and Nau, 1995, p. 804, Footnote 7).

[8] The implicit assumption is that $\lambda > 0$. If not, the two conditions are not equivalent. In our particular case, as described in Table 5, we have $\lambda(r_{m3} - \bar{r}_m) = 4.52(0.8518 - 0.6259) = 1.02 > 1$.

Let us call the price the project would have if it was traded as 'counterfactual equilibrium price'. We now illustrate a counterexample where the equilibrium value V_Z^e differs from the counterfactual equilibrium price. Let us consider project Z introduced in Table 5. What if one counterfactually assumes that Z is traded in the security market?[9] First of all, note that the introduction of the project in the security market renders the latter a complete market. It is thus evident that project Z's counterfactual equilibrium price cannot coincide with the equilibrium value $V_Z^e = 52.808$, as previously found, otherwise conditions (1)-(3) of *DIP* would be satisfied, and arbitrage opportunities would arise (which implies that the market would not be in equilibrium). This means that when the project is introduced in the security market, the market prices shift so that the market moves toward a new equilibrium. How does the resulting new equilibrium turn out to be? Intuition would tell us that the risky security's price should decrease to avoid arbitrage (given that the project dominates it), but this is not the case. It is easy to verify that, to avoid condition (3) of *DIP* and achieve an equilibrium, the risky security's price must increase and project Z's equilibrium price must increase to a larger extent so as to be greater than the risky security's price.[10] Suppose the new equilibrium is as represented in Table 7. The (counterfactual) equilibrium price of project Z is 121.57 and the price of the risky security is now 65.76. The market is now complete and arbitrage is not possible. The counterfactual equilibrium price of the project differs from the equilibrium value of the project (121.57 ≠ 52.808). It can be concluded that the equilibrium value, given by Equation (40), is not the price that the project would have if it was traded in the market. Contrary to the equilibrium value, the counterfactual equilibrium price is rational by definition, in the sense that arbitrage is not possible in the resulting equilibrium. This means that the counterfactual equilibrium price is obviously the correct value of the project.

Table 7: Project *Z* is Traded in the Market (First Equilibrium)

	Project is Traded in the Market (1 Share)	Risky Security (3 million Shares)	Risk-Free Security	Market (000,000)	Probability
F	$\begin{cases} 98 \\ 71 \\ 350 \end{cases}$	$\begin{cases} 98 \\ 71 \\ 100 \end{cases}$	$\begin{cases} 120 \\ 120 \\ 120 \end{cases}$	$\begin{cases} 294.000098 \\ 213.000071 \\ 300.000350 \end{cases}$	$\begin{cases} 0.3 \\ 0.4 \\ 0.3 \end{cases}$
Price	121.57	65.76	90	197.28	–
$\bar{r}$ (%)	33.91	33.51	33.33	33.51	–
NPV	0	0	0	0	–

One might think that, for valuation to be correct, one should replace the equilibrium value with the counterfactual equilibrium price. Unfortunately, the counterfactual equilibrium price

9 This assumption is equivalent to the assumption that a security with the same payoff as project Z is traded in the market.

10 This result holds regardless of the number of shares of project Z (or of the security having the same payoff as Z) that are traded in the market.

Table 8: Project Z is Traded in the Market (Second Equilibrium)					
	Project is Traded in the Market (1 Share)	**Risky Security (3 million Shares)**	**Risk-Free Security**	**Market (000,000)**	**Probability**
F	$\begin{cases} 98 \\ 71 \\ 350 \end{cases}$	$\begin{cases} 98 \\ 71 \\ 100 \end{cases}$	$\begin{cases} 120 \\ 120 \\ 120 \end{cases}$	$\begin{cases} 294.000098 \\ 213.000071 \\ 300.000350 \end{cases}$	$\begin{cases} 0.3 \\ 0.4 \\ 0.3 \end{cases}$
Price	76.197	58	90	174	–
$\overline{r}$ (%)	113.65	51.37	33.33	51.37	–
NPV	0	0	0	0	–

cannot be univocally determined. Table 8 shows another possible equilibrium for the market where project Z is traded. The equilibrium counterfactual price in this second equilibrium is equal to 76.197, which not only conflicts with the equilibrium value of the project, but also differs from the counterfactual equilibrium price previously obtained. Which one of the two counterfactual equilibrium prices is the one to be used for valuation? There is no answer to it, because there is no way of anticipating how equilibrium would be reached from a disequilibrium situation. That is, one cannot compute *ex ante* the equilibrium price that the project would have if it was traded in the security market. However, from a practical point of view, one may collect statistical data and make an *ex ante* estimation of the most probable equilibrium the market would reach. In this case, the estimated counterfactual equilibrium price could be taken as the correct project value.[11]

Remark 7: *Proposition 11* just gives us the reason why the equilibrium value may sometimes turn out to be incorrect. The correct value measuring increase in shareholders' wealth is indeed given by the equilibrium value if the market is complete and in equilibrium. Problems in project valuation arise only when the market is not complete and Condition (34) holds.[12] In this case, equilibrium value and counterfactual equilibrium price are not equal. A project's equilibrium value is therefore reliable only if the market is complete; in this case it does represent the (counterfactual) equilibrium price that the project would actually have if it was traded.

Conclusion

The CAPM is a theoretical model aimed at valuing financial assets in a security market under the assumption that the market is in equilibrium. As widely known, the CAPM may also be used as a decision criterion: an investment is worth undertaking if and only if the investment's

[11] From a theoretical point of view, upper and lower bounds can be computed for the counterfactual equilibrium price (Smith and Nau, 1995), but whenever the cost is greater than the lower limit and smaller than the upper limit, the 'optimal strategy is unclear' (Smith and Nau, 1995, p. 805), and decision is not possible (a further analysis must be conducted to reach a single estimated value).

[12] It is worth reminding that if the market is complete and in equilibrium, Condition (34) may not hold (given that the equivalent condition (3) of *DIP* may not hold). Conversely, if the market is not complete and in equilibrium, Condition (34) may hold, as we have seen.

expected rate of return is greater than the (cost-based) risk-adjusted cost of capital (Rubinstein, 1973). However, the role of this simple criterion has not been thoroughly investigated, so that errors and misunderstanding often arise in financial textbooks and papers, where the CAPM is incorporated in the NPV criterion in an unclear way, with no explicit indication of:

- The way it should be computed (use of disequilibrium data versus equilibrium data).
- The purpose it serves (decision or valuation).
- The relation NFV (excess return) bears to present value.

This paper, focusing on accept-reject situations and one-period projects, aims at providing a clarification of these issues. In particular, it shows that:

- From the CAPM, four decision rules are validly deducted: the disequilibrium NPV, the equilibrium NFV, the equilibrium NPV, and the disequilibrium NFV. All of them may be interchangeably used for decision making.
- While logically impeccable as decision tools, the disequilibrium NPV (equilibrium NFV) may lead to incorrect decisions if decision makers facing a portfolio of several projects (or a project composed of several subprojects) separately compute each project's NPV (NFV) and then sum the values obtained. The correct procedure is to sum the cash flows of the projects and then compute the disequilibrium NPV (equilibrium NFV).
- Only the equilibrium NPV and the disequilibrium NFV are additive, which means that they may be used for valuation purposes. The other two are not valuation tools, because they are nonadditive.
- The deduction of the disequilibrium NPV (equilibrium NFV) from the CAPM assumptions is possible because the decision problem is shaped as 'undertake *Z*/do not undertake *Z*'. If the problem is reframed in a more general way as 'undertake *Z*/ undertake *Y*', the two nonadditive decision rules may not be deducted from the CAPM assumptions.
- Even if the market is in equilibrium, the project's equilibrium NPV and disequilibrium NFV lead to an incorrect valuation whenever they are decreasing functions with respect to the end-of-period cash flow in some state of nature (which implies that the security market is incomplete). This result is the capital-budgeting equivalent of Dybvig and Ingersoll's (1982) result, which they find under the assumption of a complete market.
- If the above stated condition holds, the correct value would be given by the (counterfactual) equilibrium price that the project would have if it was traded in the security market. Unfortunately, this price is not univocally determined *ex ante* and one can only rely on an estimated equilibrium price based on exogenous data of the market.

A by-product of the results obtained is that additivity is not sufficient to guarantee rational decision making. Hence, the use of NPV or NFV for decision making is legitimate, but the discount rates should not be drawn from the CAPM: neither the equilibrium required rates of return nor the disequilibrium rates of return are reliable, either for decision or for valuation.❖

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